Options and Mathematics: Lecture 12

November 20, 2020

Binomial price of American derivatives

Summary of some important formulas

$$S(0) = S_0 > 0, \quad S(t) = \begin{cases} S(t-1)e^u & \text{with probability } p \\ S(t-1)e^d & \text{with probability } 1-p \end{cases},$$
$$B(t) = B_0 e^{rt}, \quad t \in \mathcal{I} = \{1, \dots, N\},$$

where $0 < p < 1$, $d < r < u$.

$$q_u = q, \quad q_d = 1 - q, \quad \text{where} \quad q = \frac{e^r - e^d}{e^u - e^d} \in (0, 1).$$

The value at time t of the (predictale) portfolio $\{h_S(t), h_B(t)\}_{t \in \mathcal{I}}$ is

$$V(t) = h_S(t)S(t) + h_B(t)B(t)$$

The portfolio process generates the **cash flow** $\{C(t), t = 0, \dots, N-1\}$ if

$$h_S(t)S(t-1) + h_B(t)B(t-1) = V(t-1) - C(t-1), \quad t \in \mathcal{I};$$

since $(h_S(1), h_B(1)) = (h_S(0), h_B(0))$, then $C(0) = 0$.

Recurrence formula for binomial price of European derivatives

$\Pi_Y(N) = Y, \quad \text{and} \quad \Pi_Y(t) = e^{-r}[q_u \Pi_Y^u(t+1) + q_d \Pi_Y^d(t+1)], \quad \text{for } t \in \{0, \dots, N-1\}.$

Binomial price of American derivatives

In contrast to European derivatives, American derivatives can be exercised at any time prior or including the expiration date.

Let $Y(t)$ be the pay-off of an American derivative exercised at time t, also called **intrinsic value** of the American derivative.

We assume that $t \in \mathcal{I} = \{1, \ldots, N\}$, while $t = 0$ is the present time.

Let $S(t)$ be the binomial stock price of the underlying stock at time t. We restrict ourselves to **standard American derivatives**, which means that

$$Y(t) = g(S(t)), \quad t \in \{0, 1, \ldots, N\},$$

where $g : (0, \infty) \to \mathbb{R}$ is the pay-off function of the derivative

For example, $g(z) = (z - K)_+$ for American call options and $g(z) = (K - z)_+$ for American put options, where $(z)_+ = \max(0, z)$ and K is the strike price of the option.

The initial pay-off $Y(0)$ is also denoted by Y_0; as it depends only on the initial price $S_0 = S(0)$ of the stock, the value of Y_0 is known.

For $t = 1, \ldots, N$, $Y(t)$ is path-dependent, namely,

$$Y(t) = Y(t, x_1, \ldots, x_t) = g(S(t, x_1, \ldots, x_t)).$$

Out first goal is to introduce a reasonable definition for the binomial fair price of the American derivative with intrinsic value $Y(t)$ and maturity $T = N$.

We denote this price by $\widehat{\Pi}_Y(t)$, while $\Pi_Y(t)$ denotes the binomial price of the corresponding European derivative with pay-off $Y(N) = g(S(N))$ at maturity $T = N$.

Any meaningful definition of fair price for American derivatives must satisfy the following:

(a) $\widehat{\Pi}_Y(N) = Y(N)$,

(b) $\widehat{\Pi}_Y(t) \geq Y(t)$

(c) $\widehat{\Pi}_Y(t) \geq \Pi_Y(t)$.

Property (a) fixes the price of the American derivative at time N.

Due to (b) and (c), a reasonable definition for the fair price of the American derivative at time $t = N - 1$ is

$$\widehat{\Pi}_Y(N-1) = \max(Y(N-1), \Pi_Y(N-1)).$$

Using the recurrence formula for the binomial price of European derivatives, we have

$$\Pi_Y(N-1) = e^{-r}(q_u \Pi_Y^u(N) + q_d \Pi_Y^d(N)) = e^{-r}(q_u \widehat{\Pi}_Y^u(N) + q_d \widehat{\Pi}_Y^d(N)),$$

where for the second equality we used that $\widehat{\Pi}_Y(N) = \Pi_Y(N)$.

Hence

$$\widehat{\Pi}_Y(N-1) = \max[Y(N-1), e^{-r}(q_u \widehat{\Pi}_Y^u(N) + q_d \widehat{\Pi}_Y^d(N))].$$

This suggests to introduce the following definition:

Definition 4.1

The binomial (fair) price $\widehat{\Pi}_Y(t)$ of the standard American derivative with intrinsic value $Y(t) = g(S(t))$ at time $t \in \{0, 1, \dots, N\}$ is defined by the recurrence formula

$$\boxed{\widehat{\Pi}_Y(N) = Y(N)}$$

and for $t \in \{0, \dots, N-1\}$

$$\boxed{\widehat{\Pi}_Y(t) = \max[Y(t), e^{-r}(q_u \widehat{\Pi}_Y^u(t+1) + q_d \widehat{\Pi}_Y^d(t+1))]}$$

Exercise 4.1 [?] Why did we not define the binomial fair price of the American derivative as $\widehat{\Pi}_Y(t) = \max(Y(t), \Pi_Y(t))$?

Theorem 4.1

Let $\widehat{\Pi}_Y(t)$, $t \in \{0, \dots, N\}$, denote the binomial price of the standard American derivative with intrinsic value $Y(t) = g(S(t))$ and maturity $T = N$.

Let $\Pi_Y(t)$ denote the binomial price of the corresponding European derivative with pay-off $Y(N) = g(S(N))$ at maturity N. The following holds:

(a) $\widehat{\Pi}_Y(t)$ is a deterministic function of $S(t)$.

(b) $\widehat{\Pi}_Y(t) \geq \Pi_Y(t)$, for all $t \in \{0, \dots, N\}$.

(c) Assuming $r \geq 0$, the American call and the European call with the same strike and maturity have the same binomial price.

Definition 4.2

An **optimal exercise time** for the American derivative with intrinsic value $Y(t)$ in a binomial market is a time $t \in \mathcal{I}$ such that $Y(t) > 0$ and $\widehat{\Pi}_Y(t) = Y(t)$.

Example of American put option

Let the strike price $K = 3/4$, and so

$$Y(t) = \left(\frac{3}{4} - S(t)\right)_+ .$$

Suppose $N = 2$ periods and

$$e^u = \frac{7}{4}, \quad e^d = \frac{1}{2}, \quad e^r = \frac{9}{8}, \quad S(0) = 1$$

so that

$$q_u = q_d = 1/2$$

The binomial tree for the stock price $S(t)$ and the intrinsic value $Y(t)$ is

$$\begin{tikzcd}
 & & S(2) = \tfrac{49}{16} \\
 & S(1) = \tfrac{7}{4} \arrow[ur, "u"] \arrow[dr, "d"] & \\
S_0 = 1 \arrow[ur, "u"] \arrow[dr, "d"] & & S(2) = \tfrac{7}{8} \\
 & S(1) = \tfrac{1}{2} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & S(2) = \tfrac{1}{4}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & Y(2) = 0 = \widehat{\Pi}_Y(2) \\
 & Y(1) = 0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
Y_0 = 0 \arrow[ur, "u"] \arrow[dr, "d"] & & Y(2) = 0 = \widehat{\Pi}_Y(2) \\
 & Y(1) = \tfrac{1}{4} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & Y(2) = \tfrac{1}{2} = \widehat{\Pi}_Y(2)
\end{tikzcd}$$

Now we compute the binomial price of the American put option at the times $t = 0, 1$.

If the stock price goes up at time $t = 1$ we have

$$\begin{aligned}
\widehat{\Pi}_Y(1,u) &= \max\left[Y(1,u), e^{-r}(q_u\widehat{\Pi}_Y^u(2,u) + q_d\widehat{\Pi}_Y^d(2,u))\right] \\
&= \max\left[Y(1,u), e^{-r}(q_u\widehat{\Pi}_Y(2,u,u) + q_d\widehat{\Pi}_Y(2,u,d))\right] \\
&= \max\left[0, \frac{8}{9}\left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 0\right)\right] = 0.
\end{aligned}$$

This of course has to be expected, since when the stock price goes up at time $t = 1$ the American put has no chance to expire in the money, hence it is worth nothing.

If the stock price goes down at time $t = 1$ we have

$$\begin{aligned}
\widehat{\Pi}_Y(1,d) &= \max\left[Y(1,d), e^{-r}(q_u\widehat{\Pi}_Y^u(2,d) + q_d\widehat{\Pi}_Y^d(2,d))\right] \\
&= \max\left[Y(1,d), e^{-r}(q_u\widehat{\Pi}_Y(2,d,u) + q_d\widehat{\Pi}_Y(2,d,d))\right] \\
&= \max\left[\frac{1}{4}, \frac{8}{9}\left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot\frac{1}{2}\right)\right] = \max\left[\frac{1}{4}, \frac{2}{9}\right] = \frac{1}{4}.
\end{aligned}$$

At time $t = 0$ we find

$$\widehat{\Pi}_Y(0) = \max\left[Y(0), e^{-r}(q_u\widehat{\Pi}_Y^u(1) + q_d\widehat{\Pi}_Y^d(1))\right] = \max\left[0, \frac{8}{9}\left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot\frac{1}{4}\right)\right] = \frac{1}{9}.$$

Hence the binomial price of the American put along the different paths of the stock price is as follows:

$$
\begin{tikzcd}
 & & \widehat{\Pi}_Y(2) = 0 \\
 & \widehat{\Pi}_Y(1) = 0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
\widehat{\Pi}_Y(0) = \frac{1}{9} \arrow[ur, "u"] \arrow[dr, "d"'] & & \widehat{\Pi}_Y(2) = 0 \\
 & \boxed{\widehat{\Pi}_Y(1) = \dfrac{1}{4}} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & \widehat{\Pi}_Y(2) = \frac{1}{2}
\end{tikzcd}
$$

We observe that if and only if the stock price goes down at time $t = 1$ the intrinsic value of the American put is positive and equals its binomial price prior to maturity, hence in this case (and only in this case) the earlier exercise of the put is optimal.