Options and Mathematics: Lecture 17

December 1, 2020

Probabilistic formulation of the binomial model

We want to reformulate the binomial options pricing model using the language of probability theory.

The binomial stock price can be interpreted as a stochastic process defined on the N-coin toss probability space $(\Omega_N, \mathbb{P}_p)$.

Recall that, for a given $0 < p < 1$, the binomial stock price at time $t = 0$ is given by the deterministic constant $S(0) = S_0 > 0$, while at future times we have

$$S(t) = \left\{ \begin{array}{ll} S(t-1)e^u & \text{with probability } p \\ S(t-1)e^d & \text{with probability } 1-p, \end{array} \right. , \quad t \in \mathcal{I} = \{1, \ldots, N\},$$

where $u > d$.

Now, for $t \in \mathcal{I}$, consider the random variable

$$X_t : \Omega_N \to \mathbb{R}, \quad X_t(\omega) = \begin{cases} 1, & \text{if the } t^{th} \text{ toss in } \omega \text{ is } H \\ -1, & \text{if the } t^{th} \text{ toss in } \omega \text{ is } T \end{cases}.$$

We can write $S(t)$ as

$$S(t) = S(t-1)\exp(\frac{u+d}{2} + \frac{u-d}{2}X_t) = S(t-2)\exp(2\frac{u+d}{2} + \frac{u-d}{2}(X_t + X_{t-1})$$
$$= \cdots = S_0 \exp(t\frac{u+d}{2} + \frac{u-d}{2}M_t) : \Omega_N \to \mathbb{R},$$

where $M_t = X_1 + \cdots + X_t$.

Thus $S(t)$ is a random variable and therefore $\{S(t)\}_{t=0,\dots,N}$ is a N-period stochastic process on the N-coin toss probability space.

For each $\omega \in \Omega_N$, the vector $(S(0,\omega), \dots, S(N,\omega))$ is a path for the stock price.

Letting $M_0 = 0$ we obtain that $\{M_t\}_{t=0,\dots,N}$ is a random walk and $\{S(t)\}_{t=0,\dots,N}$ is measurable, but not predictable, with respect to $\{M_t\}_{t=0,\dots,N}$.

The value at time t of the risk-free asset is the deterministic function of time $B(t) = B_0 \exp(rt)$, where r is the (constant) risk-free rate of the money market and $B_0 = B(0) > 0$ is the initial value of the risk-free asset.

Recall that $S^*(t) = e^{-rt}S(t)$ is called the discounted price of the stock.

Theorem 5.4

If $r \notin (d,u)$, there is no probability measure $\mathbb{P}_p$ on the sample space Ω_N such that the discounted stock price process $\{S^*(t)\}_{t=0,\dots,N}$ is a martingale. For $r \in (d,u)$, $\{S^*(t)\}_{t=0,\dots,N}$ is a martingale with respect to the probability measure $\mathbb{P}_p$ if and only if $p = q$, where

$$q = \frac{e^r - e^d}{e^u - e^d}.$$

Proof. By definition, $\{S^*(t)\}_{t=0,\dots,N}$ is a martingale if and only if

$$\mathbb{E}_p[e^{-rt}S(t)|S^*(0),\dots,S^*(t-1)] = e^{-r(t-1)}S(t-1), \quad \text{for all } t \in \mathcal{I}.$$

As r is constant, taking the expectation conditional to $S^*(0),\dots,S^*(t-1)$ is the same as taking the expectation conditional to $S(0),\dots,S(t-1))$; hence the above equation is equivalent to

$$\mathbb{E}_p[S(t)|S(0),\dots,S(t-1)] = e^r S(t-1), \quad \text{for all } t \in \mathcal{I},$$

where we canceled out a factor e^{-rt} in both sides of the equation. Moreover

$$\begin{aligned}\mathbb{E}_p[S(t)|S(0),\dots,S(t-1)] &= \mathbb{E}_p[\frac{S(t)}{S(t-1)}S(t-1)|S(0),\dots,S(t-1)] \\ &= S(t-1)\mathbb{E}_p[\frac{S(t)}{S(t-1)}|S(0),\dots,S(t-1)],\end{aligned}$$

where we used that $S(t-1)$ is measurable with respect to the conditioning variables and thus it can be taken out from the conditional expectation (see property 5 in Theorem 5.3). As

$$S(t)/S(t-1) = \begin{cases} e^u & \text{with prob. } p \\ e^d & \text{with prob. } 1-p \end{cases}$$

is independent of $S(0),\dots,S(t-1)$, then by Theorem 5.3(2) we have

$$\mathbb{E}_p[\frac{S(t)}{S(t-1)}|S(0),\dots,S(t-1)] = \mathbb{E}_p[\frac{S(t)}{S(t-1)}] = e^u p + e^d(1-p).$$

Hence $S^*(t)$ is a martingale if and only if $e^u p + e^d(1-p) = e^r$. Solving in p we find $p = q$. Moreover $q \in (0,1)$ holds if and only if $r \in (d,u)$. $\square$

Remarks

- Due to the previous theorem, $\mathbb{P}_q$ is called **martingale probability measure**. Moreover we can reformulate Theorem 2.4 as follows:

  A binomial market is free of self-financing arbitrages if and only if there exists a martingale probability measure.

  This result holds not only for the binomial model discussed in this text, but also for any discrete (or even continuum) market model and is named **first fundamental theorem of asset pricing**.

- Since martingales have constant expectation, we obtain the important result $\mathbb{E}_q[S^*(t)] = \mathbb{E}[S^*(0)]$, or equivalently,

  $$\boxed{\mathbb{E}_q[S(t)] = S_0 e^{rt}}$$

  Thus in the martingale probability measure one expects the same return on the stock as on the risk-free asset. For this reason, $\mathbb{P}_q$ is also called **risk-neutral probability**.

Risk-neutral price of European derivatives

The value of a portfolio position (h_S, h_B) invested on h_S shares of the stock and h_B shares of the risk-free asset is the stochastic process $\{V(t)\}_{t=0,\dots,N}$ on $(\Omega_N, \mathbb{P}_p)$ given by

$$V(t) = h_B B(t) + h_S S(t) \,:\, \Omega_N \to \mathbb{R}, \quad t = 0, \dots, N.$$

Note that $V(0) = h_S S_0 + h_B B_0$ is a deterministic constant.

If we change the portfolio position depending on the price of the stock (i.e., depending on $\omega \in \Omega_N$), then we get a portfolio (stochastic) process $\{(h_S(t), h_B(t))\}_{t=0,\dots,N}$.

Recall that $(h_S(t), h_B(t))$ corresponds to the portfolio position held in the interval $(t-1, t]$ and, by convention, $(h_S(0), h_B(0)) = (h_S(1), h_B(1))$.

The portfolio process is predictable if $h_S(t), h_B(t)$ is a deterministic function of $S(0), \ldots, S(t-1)$, which means that the stochastic process $\{h_S(t), h_B(t)\}_{t=0,\ldots,N}$ is predictable from the stochastic process $\{S(t)\}_{t=0,\ldots,N}$ in the sense of Definition 5.15

The value $\{V(t)\}_{t=0,\ldots,N}$ of the portfolio process is the stochastic process given by

$$V(t) = h_B(t)B(t) + h_S(t)S(t) \,:\, \Omega_N \to \mathbb{R}, \quad t = 0, \ldots, N.$$

A portfolio process $\{(h_S(t), h_B(t))\}_{t=0,\ldots,N}$ is said to be self-financing if

$$V(t-1) = h_B(t)B(t-1) + h_S(t)S(t-1), \quad t \in \mathcal{I}.$$

In Theorem 2.2 we have shown that the value at time $t = 0$ of self-financing processes is determined by the value at time N through the formula

$$V(0) = e^{-rN} \sum_{x \in \{u,d\}^N} (q_u)^{N_u(x)} (q_d)^{N_d(x)} V(N, x)$$

This result can be written in terms of the expectation in the martingale probability measure as

$$V(0) = e^{-rN} \mathbb{E}_q[V(N)].$$

Now, let $Y : \Omega_N \to \mathbb{R}$ be a random variable and consider the European derivative with pay-off Y at time of maturity $T = N$.

The binomial price of this derivative at time $t = 0$ equals the value at time $t = 0$ of any self-financing hedging portfolio.

Hence replacing $V(N) = Y$ in the formula above we obtain the following fundamental result.

Theorem 5.5

The binomial price at time $t = 0$ of the European derivative with pay-off Y at maturity $T = N$ can be rewritten as

$$\boxed{\Pi_Y(0) = e^{-rN}\mathbb{E}_q[Y]}$$

or equivalently

$$\boxed{\Pi_Y(0) = \mathbb{E}_q[Y^*]}$$

where Y^* is the discounted value of the pay-off.

Remark

As proved in Theorem 3.3, any European derivative in the binomial market can be hedged by a self-financing portfolio invested in the underlying stock and the risk-free asset. For this reason the binomial market is called a **complete market**. In fact, the **second fundamental theorem of asset pricing** states that market completeness is equivalent to the uniqueness of the risk-neutral probability measure. An arbitrage free market is said to be **incomplete** if the risk-neutral measure is not unique. When the market is incomplete the price of European derivatives is not uniquely defined and moreover there exist European derivatives which cannot be hedged by self-financing portfolios.

Risk-neutral price of American derivatives

Consider now a standard American derivative with intrinsic value $Y(t)$, $t \in \mathcal{I}$, and maturity $T = N$.

We have defined the binomial fair price $\widehat{\Pi}_Y(t)$ of this derivative using the recurrence formula

$$\widehat{\Pi}_Y(N) = Y(N), \quad \max[Y(t), e^{-r}(q_u\widehat{\Pi}^u_Y(t+1)+q_d\widehat{\Pi}^d_Y(t+1))], \quad t = 0, \dots, N-1.$$

Our next purpose is to write the definition of binomial price of American derivative in the form of a risk-neutral pricing formula.

Suppose first that the seller knows at time $t = 0$ that the buyer of the American derivative will exercise at time $\tau \in \mathcal{I}$.

In this case the discounted (at time $t = 0$) value of the pay-off is $e^{-r\tau}Y(\tau)$ and it is therefore reasonable to define the fair price of the American derivative at time $t = 0$ as $\widehat{\Pi}_Y(0) = \mathbb{E}_q[e^{-r\tau}Y(\tau)]$.

Considering that in the real world the seller does not know at which time the American derivative will be exercised, and being the exercise time any possible $\tau \in \mathcal{I}$, one may (erroneously) think that the fair value of the derivative at time $t = 0$ should be $\widehat{\Pi}_Y(0) = \max_{\tau\in\mathcal{I}} \mathbb{E}_q[e^{-r\tau}Y(\tau)]$.

However this definition is unfair for two reasons:

(1) it does not take into account that the exercise time is a random variable and

(2) the decision to exercise at time $\tau = k \in \mathcal{I}$ is taken only using the information available at time k (and not on the uncertain future), which is contained in the stock prices $S(1), \dots, S(k)$.

Thus the exercise time is a random variable in the space Q defined as follows:

$$Q = \left\{ \begin{array}{ll} \text{random variables } X \text{ such that:} & (i)\ \mathrm{Im}(X) = \{1, \dots, N\} \\ & (ii) \text{ the event } X = k \text{ is resolved} \\ & \text{by the random variables } S(1), \dots, S(k) \\ & \text{for all } k = 1, \dots, N. \end{array} \right\}$$

The discussion above leads to the following definition.

Definition 5.17

The (binomial) risk-neutral price at time $t = 0$ of the American derivative with intrinsic value $\{Y(t)\}_{t \in \mathcal{I}}$ and maturity $T = N$ is defined as

$$\widehat{\Pi}_Y(0) = \max_{\tau \in Q} \mathbb{E}_q[e^{-r\tau} Y(\tau)],$$

where Q is the set of random variables defined above.