Options and Mathematics: Lecture 20

December 3, 2020

Black-Scholes markets

Definition 6.7 Let $\{W(t)\}_{t\geq 0}$ be a Brownian motion, $\alpha \in \mathbb{R}$, $\sigma > 0$ and $S_0 > 0$ be constants. The positive stochastic process $\{S(t)\}_{t\geq 0}$ given by

$$\boxed{S(t) = S_0 e^{\alpha t + \sigma W(t)}}$$

is called a **geometric Brownian motion** (GBM).

We shall use geometric Brownian motions to model the dynamics of stock prices in the time-continuum case.

More precisely, a **Black-Scholes market** is a market that consists of a stock with price given by a GBM, and a risk-free asset with constant interest rate r; in particular, the value of the risk-free asset at time t is given by

$$B(t) = B_0 e^{rt} \quad B_0 = B(0) > 0$$

We assume throughout that $t \in [0, T]$, where $T > 0$ could be for instance the time of maturity of a financial derivative on the stock.

The probability $\mathbb{P}$ with respect to which $\{W(t)\}_{t\geq 0}$ is Brownian motion is the **physical** (or **real-world**) **probability** of the Black-Scholes market.

α is the **instantaneous mean of log-return**, σ is the **instantaneous volatility** and σ^2 is the **instantaneous variance** of the geometric Brownian motion.

To justify this terminology we now show that α and σ satisfy the analogous interpretations as in the binomial model. Namely, for all $t \in [0, T]$ and $h > 0$ such that $t + h \leq T$ we have

$$\alpha = \frac{1}{h}\mathbb{E}[\log S(t+h) - \log S(t)], \quad \sigma^2 = \frac{1}{h}\mathrm{Var}[\log S(t+h) - \log S(t)].$$

In fact, since $W(t) \in \mathcal{N}(0, t)$, we have

$$\begin{aligned}\mathbb{E}[\log S(t+h) - \log S(t)] &= \mathbb{E}[\alpha h + \sigma W(t+h) - \sigma W(t)] \\ &= \alpha h + \sigma(\mathbb{E}[W(t+h)] - \mathbb{E}[W(t)]) = \alpha h.\end{aligned}$$

Similarly

$$\begin{aligned}\mathrm{Var}[\log S(t+h) - \log S(t)] &= \mathrm{Var}[\alpha h + \sigma W(t+h) - \sigma W(t)] \\ &= \sigma^2 \mathrm{Var}[W(t+h) - W(t)] = \sigma^2 h,\end{aligned}$$

where we used that the increment $W(t+h) - W(t)$ belongs to $\mathcal{N}(0, h)$.

Next we derive the density function of the geometric Brownian motion.

Theorem 6.10

The density of the random variable $S(t)$ is given by

$$f_{S(t)}(x) = \frac{H(x)}{\sqrt{2\pi\sigma^2 t}}\frac{1}{x}\exp\left(-\frac{(\log x - \log S(0) - \alpha t)^2}{2\sigma^2 t}\right),$$

where $H(x)$ is the **Heaviside** function.

Proof. The density of $S(t)$ is given by

$$f_{S(t)}(x) = \frac{d}{dx} F_{S(t)}(x),$$

where $F_{S(t)}$ is the distribution of $S(t)$, i.e.,

$$F_{S(t)}(x) = \mathbb{P}(S(t) \leq x).$$

Clearly, $f_{S(t)}(x) = F_{S(t)}(x) = 0$, for $x \leq 0$. For $x > 0$ we use that

$$S(t) \leq x \quad \text{if and only if} \quad W(t) \leq \frac{1}{\sigma}\left(\log \frac{x}{S(0)} - \alpha t\right) := A(x).$$

Thus, using $W(t) \in \mathcal{N}(0, t)$,

$$\mathbb{P}(S(t) \leq x) = \mathbb{P}(-\infty < W(t) \leq A(x)) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{A(x)} e^{-\frac{y^2}{2t}}\, dy,$$

where for the second equality we used that $W(t) \in \mathcal{N}(0, t)$. Hence

$$f_{S(t)}(x) = \frac{d}{dx}\left(\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{A(x)} e^{-\frac{y^2}{2t}}\, dy\right) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{A(x)^2}{2t}} \frac{dA(x)}{dx},$$

for $x > 0$, that is

$$f_{S(t)}(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \frac{1}{x} \exp\left\{-\frac{(\log x - \log S(0) - \alpha t)^2}{2\sigma^2 t}\right\}, \quad x > 0.$$

The proof is complete. $\square$

The risk-neutral pricing formula in Black-Scholes markets

The purpose of this section is to introduce the definition of Black-Scholes price of European derivatives from a probability theory point of view.

Recall that the probabilistic interpretation of the binomial price is encoded in the risk-neutral pricing formula.

Our goal is to derive a similar risk-neutral pricing formula (at time $t = 0$) for the time-continuum Black-Scholes model.

Motivated by the approach for the binomial model, we first look for a probability measure in which the the discounted stock price in Black-Scholes markets is a martingale (martingale probability measure).

It is natural to seek such martingale probability within the class of Girsanov probabilities $\{\mathbb{P}_\theta\}_{\theta\in\mathbb{R}}$; recall that $\mathbb{P}_\theta$ is equivalent to the physical probability $\mathbb{P} = \mathbb{P}_0$ for all $\theta \in \mathbb{R}$.

We shall need the form of the density function of the geometric Brownian motion in the probability measure $\mathbb{P}_\theta$.

Theorem 6.11

Let $\theta \in \mathbb{R}$ and $\{W(t)\}_{t\geq 0}$ be a $\mathbb{P}$-Brownian motion. The geometric Brownian motion has the following density in the probability measure $\mathbb{P}_\theta$:

$$f^{(\theta)}_{S(t)}(x) = \frac{H(x)}{\sqrt{2\pi\sigma^2 t}}\frac{1}{x}\exp\left(-\frac{(\log x - \log S_0 - (\alpha - \theta\sigma)t)^2}{2\sigma^2 t}\right).$$

Proof. Since

$$S(t) = S_0 e^{\alpha t + \sigma W(t)} = S_0 e^{(\alpha-\theta\sigma)t + \sigma W^{(\theta)}(t)}, \quad W^{(\theta)}(t) = W(t) + \theta t$$

and since $\{W^{(\theta)}(t)\}_{t\geq 0}$ is a Brownian motion in the probability measure $\mathbb{P}_\theta$ (by Girsanov's Theorem), then the density $f^{(\theta)}_{S(t)}$ is the same as in Theorem 6.10 with α replaced by $\alpha - \theta\sigma$. $\square$

Let $\mathbb{E}_\theta[\cdot]$ denote the expectation in the measure $\mathbb{P}_\theta$. Recall that martingales have constant expectation. Hence in the martingale (or risk-neutral) probability measure the expectation of the discounted value of the stock must be constant, i.e., $\mathbb{E}_\theta[S(t)] = S_0 e^{rt}$. We now show that this condition alone suffices to single out a unique possible value of θ, namely

The identity $\mathbb{E}_\theta[S(t)] = S_0 e^{rt}$ *holds if and only if* $\theta = q$, *where*

$$q = \frac{\alpha - r}{\sigma} + \frac{\sigma}{2}.$$

In fact, using the density of $S(t)$ in the probability $\mathbb{P}_\theta$ we have

$$\mathbb{E}_\theta[S(t)] = \int_{\mathbb{R}} x f^{(\theta)}_{S(t)}(x)\, dx = \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_0^\infty \exp\left(-\frac{(\log x - \log S_0 - (\alpha - \theta\sigma)t)^2}{2\sigma^2 t}\right) dx.$$

With the change of variable

$$y = \frac{\log x - \log S_0 - (\alpha - \theta\sigma)t}{\sigma\sqrt{t}}, \quad dx = x\sigma\sqrt{t}\, dy,$$

we obtain

$$\mathbb{E}_\theta[S(t)] = \frac{S_0}{\sqrt{2\pi}} e^{(\alpha-\theta\sigma)t} \int_{\mathbb{R}} e^{-\frac{y^2}{2} + \sigma\sqrt{t}y}\, dy = S_0 e^{(\alpha - \theta\sigma + \frac{\sigma^2}{2})t} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{(y+\sigma\sqrt{t})^2}{2}}\, dy.$$

As $\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, dx = 1$, the claim follows.

Even though the validity of $\mathbb{E}_\theta[S(t)] = S_0 e^{rt}$ is only necessary for the discounted geometric Brownian motion to be a martingale, one can show that the following result holds.

Theorem 6.12 The discounted value of the geometric Brownian motion stock price is a martingale in the Girsanov probability measure $\mathbb{P}_\theta$ if and only if $\theta = q$, where q is given as above.

The probability measure $\mathbb{P}_q$ is called the **martingale probability**, or **risk-neutral probability**, of the Black-Scholes market. Replacing $\alpha = r + q\sigma - \frac{1}{2}\sigma^2$ in the the GBM we may rewrite the stock price as

$$\boxed{S(t) = S(0)e^{(r-\frac{\sigma^2}{2})t+\sigma W^{(q)}(t)}}$$

where we recall that $W^{(q)}(t) = W(t) + qt$ and, by Girsanov's theorem, $\{W^{(q)}(t)\}_{t\geq 0}$ is a Brownian motion in the risk-neutral probability.

It follows that *in the probability measure $\mathbb{P}_q$, $S(t)$ is a geometric Brownian motion with volatility σ and mean of log return $\mu = r - \frac{1}{2}\sigma^2$.*

At this point we have all we need to define the Black-Scholes price of European derivatives at time $t = 0$ using the risk-neutral pricing formula.

Definition 6.18

The Black-Scholes price at time $t = 0$ of the European derivative with pay-off Y at maturity T is given by the risk-neutral pricing formula

$$\boxed{\Pi_Y(0) = e^{-rT}\mathbb{E}_q[Y]}$$

i.e., it equals the expected value of the discounted pay-off in the risk-neutral probability measure of the Black-Scholes market.

In the case of standard European derivatives we can use the density of the geometric Brownian motion in the risk-neutral probability measure to write the Black-Scholes price in the following integral form.

Theorem 6.13

For the standard European derivative with pay-off $Y = g(S(T))$ at maturity $T > 0$, the Black-Scholes price at time $t = 0$ can be written as

$$\boxed{\Pi_Y(0) = v_0(S_0)}$$

where S_0 is the price of the underlying stock at time $t = 0$ and $v_0 : (0,\infty) \to \mathbb{R}$ is the **pricing function** of the derivative at time $t = 0$, which is given by

$$\boxed{v_0(x) = e^{-rT} \int_{\mathbb{R}} g(xe^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}y})e^{-\frac{1}{2}y^2}\,\frac{dy}{\sqrt{2\pi}}}$$

Proof. Replacing $\theta = q$ in the density of GBM in the risk-neutral probability we obtain that the geometric Brownian motion has the following density in the probability $\mathbb{P}_q$:

$$f^{(q)}_{S(t)}(x) = \frac{H(x)}{\sqrt{2\pi\sigma^2 t}}\frac{1}{x}\exp\left(-\frac{(\log x - \log S_0 - (r - \frac{\sigma^2}{2})t)^2}{2\sigma^2 t}\right).$$

Using this density for $t = T$ in the risk-neutral pricing formula we obtain

$$\begin{aligned}\Pi_Y(0) = e^{-rT}\mathbb{E}_q[Y] = e^{-rT}\mathbb{E}_q[g(S(T))] &= \int_{\mathbb{R}} g(x) f^{(q)}_{S(T)}(x)\,dx \\ &= \frac{e^{-rT}}{\sqrt{2\pi\sigma^2 t}}\int_0^\infty \frac{g(x)}{x}\exp\left(-\frac{(\log x - \log S_0 - (r - \frac{\sigma^2}{2})t)^2}{2\sigma^2 t}\right)dx.\end{aligned}$$

With the change of variable $y = \frac{\log x - \log S_0 - (\alpha - \theta\sigma)t}{\sigma\sqrt{t}}$ we obtain

$$\Pi_Y(0) = e^{-rT}\int_{\mathbb{R}} g(S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}y})e^{-\frac{1}{2}y^2}\,\frac{dy}{\sqrt{2\pi}} = v_0(S_0),$$

as claimed. $\square$

The risk-neutral pricing formula for $t > 0$ is

$$\Pi_Y(t) = e^{-rT}\mathbb{E}_q[Y|\mathcal{F}_S(t)],$$

The right hand side of is the expectation of the discounted pay-off in the risk-neutral probability measure conditional to the information available at time t, which in a Black-Scholes market is determined by the history of the stock price up to time t.

It can be shown that in the case of the standard European derivative with pay-off $Y = g(S(T))$ and maturity T, the risk-neutral pricing formula at time $t > 0$ entails that the Black-Scholes price at time $t \in [0, T]$ can be written in the integral form

$$\boxed{\Pi_Y(t) = v(t, S(t))}$$

where

$$\boxed{v(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} g\left(xe^{(r-\frac{\sigma^2}{2})\tau}e^{\sigma\sqrt{\tau}\,y}\right) e^{-\frac{y^2}{2}}\,dy \quad \tau = T - t.}$$

Hence the pricing function $v(t, x)$ of the derivative at time t is the same as the pricing function at time $t = 0$ but with maturity T replaced by the time τ left to maturity, which is rather intuitive.

Hedging portfolio

A portfolio process $\{h_S(t), h_B(t)\}_{t\in[0,T]}$ invested in a Black-Scholes market is said to be hedging the European derivative with pay-off Y and maturity $T > 0$ if

$$V(T) = Y$$

where

$$V(t) = h_S(t)S(t) + h_B(t)B(t)$$

is the value of the portfolio process at time $t \in [0, T]$.

The portfolio process is said to be replicating the derivative if $V(t) = \Pi_Y(t)$, for all $t \in [0, T]$, where $\Pi_Y(t)$ is the Black-Scholes price of the derivative.

It can be shown that the Black-Scholes price $\Pi_Y(t)$ coincides with the value at time $t \in [0, T]$ of any self-financing portfolio processes hedging the derivative, precisely as in the binomial model. However the definition of self-financing portfolio in Black-Scholes markets requires the use of stochastic calculus and it is therefore beyond the purpose of this course.

Moreover it can be shown that in the case of standard European derivatives the portfolio process $\{(h_S(t), h_B(t))\}_{t\in[0,T]}$ given by

$$\boxed{h_S(t) = \Delta(t, S(t)), \quad \Delta(t, x) = \partial_x v(t, x)}$$

$$\boxed{h_B(t) = \frac{1}{B(t)}(\Pi_Y(t) - h_S(t)S(t))}$$

is self-financing and hedges the derivative. Here v denotes the Black-Scholes pricing function and $\partial_x v$ the partial derivative of v in the second variable (i.e., the derivative in x assuming that t is constant).

Note that the formula for $h_B(t)$ is equivalent to the replicating condition $V(t) = \Pi_Y(t)$ of the portfolio process $\{h_S(t), h_B(t)\}_{t\in[0,T]}$.