Options and Mathematics: Lecture 22

December 9, 2020

Exercises

Exercise 6.6

Let $\{W(t)\}_{t\geq 0}$ be a Brownian motion. Show that

$$\mathrm{Cov}[W(s), W(t)] = \min(s, t), \quad \text{for all } s, t \geq 0.$$

(Solution can be found in the book)

Exercise 6.7

Let $\{W(t)\}_{t\geq 0}$ be a $\mathbb{P}$-Brownian motion and $T > 0$. Given a differentiable function $\theta : (0, \infty) \to \mathbb{R}$, define

$$Z_\theta = \exp\left(-\theta(T)W(T) + \int_0^T \theta'(s)W(s)\,ds - \frac{1}{2}\int_0^T \theta^2(s)\,ds\right).$$

Show that $\mathbb{P}_\theta(A) = \mathbb{E}[Z_\theta \mathbb{I}_A]$ is a probability measure equivalent to $\mathbb{P}$.

HINT: You need Theorem 6.6:

Let $g : (0, \infty) \to \mathbb{R}$ be a differentiable function and let

$$X(t) = g(t)W(t) - \int_0^t g'(s)W(s)\,ds.$$

Then

$$X(t) \in \mathcal{N}(0, \Delta(t)), \quad \Delta(t) = \int_0^t g(s)^2\,ds.$$

Exercise 6.9

Suppose that at time $t = 0$ it is assumed that the stock price is described by a geometric Brownian motion

$$S(t) = S(0)e^{\alpha t + \sigma W(t)}$$

in the interval $[0, T]$. Given any arbitrary subinterval $[t_0, t] \subset [0, T]$ with length $\tau = t - t_0$, define the random variable

$$\sigma_\tau^2(t) = \frac{1}{h(n-1)} \sum_{i=1}^{n} (\widehat{R}_i - \widehat{R})^2,$$

where $t_0 < t_1 < t_2 < \cdots < t_n = t$ is a partition of $[t_0, t]$ with $h = t_i - t_{i-1}$ and $\widehat{R}_i, \widehat{R}$ are given by

$$\widehat{R}_i = \log S(t_i) - \log S(t_{i-1}) = \log\left(\frac{S(t_i)}{S(t_{i-1})}\right), \quad i = 1, \ldots, n.$$

$$\widehat{R}(t) = \frac{1}{n}\sum_{i=1}^{n} \widehat{R}_i = \frac{1}{n}\log\left(\frac{S(t)}{S(t_0)}\right).$$

.Show that

$$\mathbb{E}[\sigma_\tau(t)^2] = \sigma^2.$$

In other words, σ^2 is the expected value of the τ-historical variance at any time $t \in [0, T]$.

(Solution can be found in the book)

Exercise 6.12

Let $C(t, x, K, T)$ be the Black-Scholes pricing function of the call with strike K and maturity T.

Prove that

$$\lim_{\sigma \to 0^+} C(t, x, K, T) = (x - Ke^{-r\tau})_+, \quad \lim_{\sigma \to \infty} C(t, x, K, T) = x.$$

Compute also the following limits:

$$\lim_{K \to 0^+} C(t, x, K, T), \qquad \lim_{K \to +\infty} C(t, x, K, T), \qquad \lim_{\tau \to +\infty} C(t, x, K, T), \qquad \lim_{x \to 0^+} C(t, x, K, T)$$

and show that $C(t, x, K, T)$ is asymptotic to $x - Ke^{-r\tau}$ as $x \to \infty$. Compute the same limits for put options.

(Solution can be found in the book)

Exercise 6.13

A **binary** (or **digital**) call option with strike K and maturity T pays-off the buyer if and only if $S(T) > K$. If the pay-off is a fixed amount of cash L, then the binary call option is said to be "cash-settled", while if the pay-off is the stock itself then the option is said to be "physically settled". Compute the Black-Scholes price of the cash-settled binary call option and the number of shares on the stock in the self-financing hedging portfolio.

(Solution can be found in the book)

Exercise 6.15

Consider the European derivative with maturity T and pay-off Y given by

$$Y = k + S(T) \log S(T),$$

where $k > 0$ is a constant. Find the Black-Scholes price of the derivative at time $t < T$ and the self-financing hedging portfolio. Find the probability that the derivative expires in the money.

(Solution can be found in the book)

Solution 6.17

Let $K > 0$. A European style derivative on a stock with maturity $T > 0$ gives to its owner the right to choose between selling the stock for the price K at time T or paying the amount K at time T. Draw the pay-off function of the derivative. Compute the Black-Scholes price of the derivative. Show that there exists a value $K_* > 0$ of K such that the Black-Scholes price of the derivative is zero.

(Solution can be found in the book)