Options and Mathematics: Lecture 24

December 11, 2020

Standard European derivatives on a dividend-paying stock

In this llecture we compute the Black-Scholes price of standard European derivatives on a dividend-paying stock.

In a frictionless market this means that the price of the stock drops at some time $t_0 \in (0,T)$ of a quantity $D < S(t_0)$, which is deposited in the account of the shareholders.

Letting $S(t_0^-) = \lim_{t\to t_0^-} S(t)$, we then have

$$S(t_0) = S(t_0^-) - D.$$

We assume that on each of the intervals $[0,t_0)$, $[t_0,T]$, the stock price follows a geometric Brownian motion.

In the risk-neutral probability $\mathbb{P}_q$ this means that

$$\begin{aligned}
S(s) &= S(t)e^{(r-\frac{1}{2}\sigma^2)(s-t)+\sigma(W^{(q)}(s)-W^{(q)}(t))}, \quad t \in [0,t_0),\ s \in [t,t_0) \\
S(s) &= S(u)e^{(r-\frac{1}{2}\sigma^2)(s-u)+\sigma(W^{(q)}(s)-W^{(q)}(u))}, \quad u \in [t_0,T],\ s \in [u,T].
\end{aligned}$$

To simplify the discussion we consider the case in which the dividend D is expressed as percentage of the stock price just before the dividend is paid, i.e., $D = aS(t_0^-)$, for some $a \in (0, 1)$.

Note that this means that we do not know at time $t = 0$ what amount the dividend will pay at time $t_0 > 0$, but we known for sure that $D < S(t_0^-)$.

Theorem 6.18

Consider the standard European derivative with pay-off $Y = g(S(T))$ and maturity T. Let $\Pi_Y^{(a,t_0)}(0)$ be the Black-Scholes price of the derivative at time $t = 0$ assuming that the underlying stock pays the dividend $aS(t_0^-)$ at time $t_0 \in (0, T)$, where $a \in (0, 1)$. Then

$$\boxed{\Pi_Y^{(a,t_0)}(0) = v_0((1 - a)S_0)}$$

where $v_0(x)$ is the pricing function in the absence of dividends.

Exercise 6.28[?]

Use the previous result to show that the call option is less valuable at time $t = 0$ if the stock pays a dividend at time $t_0 > 0$. Give an intuitive explanation for this property.

Exercise 6.29

A standard European derivative pays the amount $Y = (S(T) - S(0))_+$ at time of maturity T. Find the Black-Scholes price $\Pi_Y(0)$ of this derivative at time $t = 0$ assuming $r > 0$ and that the underlying stock pays the dividend $(1 - e^{-rT})S(\frac{T}{2}-)$ at time $t = \frac{T}{2}$. Compute the probability of positive return for a constant portfolio which is short 1 share of the derivative and short $S(0)e^{-rT}$ shares of the risk-free asset (assume $B(0) = 1$).