# Solutions to some exercises from Dobrow

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8.19

$$
\begin{aligned}
\mathrm{E}(M_t) &= \mathrm{E}(|B_t|) = \int_{-\infty}^{\infty} |s| \cdot \mathrm{Normal}(s; 0, t)\, dt = 2\int_{0}^{\infty} s \cdot \mathrm{Normal}(s; 0, t)\, ds \\
&= \frac{2}{\sqrt{2\pi t}} \int_0^\infty s \exp\left(-\frac{1}{2t}s^2\right) ds = \frac{2}{\sqrt{2\pi t}} \int_0^\infty \exp(-u) t\, du \\
&= \sqrt{\frac{2t}{\pi}} \left[-e^{-u}\right]_0^\infty = \sqrt{\frac{2t}{\pi}}
\end{aligned}
$$

where we use the substitution $u = \frac{s^2}{2t}$ to compute the integral. Further, we have

$$\mathrm{E}(M_t^2) = \mathrm{E}(|B_t|^2) = \mathrm{E}(B_t^2) = \mathrm{Var}(B_t) = t$$

and thus

$$\mathrm{Var}(M_t) = \mathrm{E}(M_t^2) - \mathrm{E}(M_t)^2 = t - \frac{2t}{\pi}.$$

8.22 Let $z_{r,t}$ be the probability that standard Brownian motion has at least one zero i $(r, t)$. Then

$$\mathrm{P}(Z \leq z) = z_{t,z} = \frac{2}{\pi} \arccos\left(\sqrt{\frac{t}{z}}\right)$$

according to theorem 8.1.

8.25

$$
\begin{aligned}
\mathrm{P}(G_1 \geq 40) &= \mathrm{P}(G_0 e^{\mu 1 + \sigma B_1} \geq 40) = \mathrm{P}(35 e^{-0.25 + 0.4 B_1} \geq 40) \\
&= \mathrm{P}\left(-0.25 + 0.4 B_1 \geq \log(40/35)\right) = \mathrm{P}\left(B_1 \geq \frac{\log(40/35) + 0.25}{0.4}\right)
\end{aligned}
$$

Writing in R

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> 1 - pnorm((log(40/35)+0.25)/0.4)
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gives 0.1688226.

8.26 Generally we have

$$\mathrm{E}\left(\max\left(G_t - K, 0\right)\right) = G_0 e^{t(\mu+\sigma^2/2)}\,\mathrm{P}\left(B_1 > \frac{\beta - \sigma t}{\sqrt{t}}\right) - K\,\mathrm{P}\left(B_1 > \frac{\beta}{\sqrt{t}}\right)$$

where $\beta = (\log(K/G_0) - \mu t)/\sigma$. In our case,

$$\beta = (\log(K/G_0) - \mu t)/\sigma = (\log(40/35) + 0.25/2) = 0.2585314.$$

Thus,

$$\begin{aligned}
& \mathrm{E}\left(\max\left(G_t - K, 0\right)\right) \\
= \;& 35 e^{(-0.25+0.4^2/2)/2}\,\mathrm{P}\left(B_1 > \frac{0.2585314 - 0.4/2}{\sqrt{1/2}}\right) - 40\,\mathrm{P}\left(B_1 > \frac{0.2585314}{\sqrt{1/2}}\right) \\
= \;& 32.14793\,\mathrm{P}\left(B_1 > 0.0827759\right) - 40\,\mathrm{P}\left(B_1 > 0.3656196\right)
\end{aligned}$$

Computing with R

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32.14793*(1-pnorm(0.0827759)) - 40*(1-pnorm(0.3656196))
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gives 0.7205823.

8.29 Note that

$$\mathrm{E}((B_t - B_s)^3) = \mathrm{E}(B_{t-s}^3) = 0$$

because of symmetry of the Normal distribution. Thus we get

$$\begin{aligned}
& \mathrm{E}\left(B_t^3 - 3tB_t \mid B_r, 0 \le r \le s\right) \\
= \;& \mathrm{E}\left((B_t - B_s + B_s)^3 - 3t(B_t - B_s + B_s) \mid B_r, 0 \le r \le s\right) \\
= \;& \mathrm{E}\left((B_t - B_s)^3 + 3(B_t - B_s)^2 B_s + 3(B_t - B_s)B_s^2 + B_s^3 - 3t(B_t - B_s) - 3tB_s \mid B_r, 0 \le r \le s\right) \\
= \;& 3\,\mathrm{E}(B_{t-s}^2)B_s + B_s^3 - 3tB_s = 3(t-s)B_s + B_s^3 - 3tB_s = B_s^3 - 3sB_s.
\end{aligned}$$

Also

$$\mathrm{E}(|B_t^3 - 3tB_t|) \le \mathrm{E}(|B_t|^3) + 3t\,\mathrm{E}(|B_t|)$$

and both integrals used to compute these expectations can be shown to have finite values.