MVE550 2020 Lecture 6.1
Dobrow Sections 4.1, 4.2
Introduction to Branching processes

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Introduction

- Many real phenomena can be described as developing with a tree-like structure, for example
  - Growth of cells.
  - Spread of viruses or other pathogens in a population.
  - Nuclear chain reactions.
  - Spread of funny cat videos on the internet.
  - Spread of a surname over generations.
- The process with which one node gives rise to "children" can be described as random: We will assume the probabilistic properties of this process is the same for all nodes.
- We will assume all nodes are organized into *generations*.
- We are only concerned with the size of each generation.
- How large are the generations? How much does the size vary? Will the process become *extinct*?

Branching processes

A branching process is discrete Markov chain $Z_0, Z_1, \dots, Z_n, \dots$ where

- the state space is the non-negative integers
- $Z_0 = 1$
- 0 is an absorbing state
- Z_n is the sum $X_1 + X_2 + \cdots + X_{Z_{n-1}}$, where the X_j are independent random non-negative integers all with the same *offspring distribution*. In other words

$$Z_n = \sum_{i=1}^{Z_{n-1}} X_i.$$

- Connecting each of the Z_n individuals in generation n with their offspring in generation $n+1$ we get a tree illustrating the branching process.
- The offspring distribution is described by the probability vector $a = (a_0, a_1, \dots,)$ where $a_j = \Pr(X_i = j)$.
- To focus on the interesting cases we assume $a_0 > 0$ and $a_0 + a_1 < 1$.

Expected generation size

- Note that the state 0 is absorbing: This absorbtion is called *extinction*.
- We can show that all nonzero states are transient.
- Define $\mu = \mathrm{E}(X_i) = \sum_{j=0}^{\infty} j a_j$.
- Then one can show that

$$\mathrm{E}(Z_n) = \mathrm{E}\left(\sum_{i=1}^{Z_{n-1}} X_i\right) = \mathrm{E}(Z_{n-1})\,\mathrm{E}(X_i) = \mathrm{E}(Z_{n-1})\,\mu.$$

- We get directly that

$$\mathrm{E}(Z_n) = \mu^n\,\mathrm{E}(Z_1) = \mu^n$$

- We subdivide Branching processes into three types:
  - *Subcritical* if $\mu < 1$. Then $\lim_{n\to\infty} \mathrm{E}(Z_n) = 0$.
  - *Critical* if $\mu = 1$. Then $\lim_{n\to\infty} \mathrm{E}(Z_n) = 1$.
  - *Supercritical* if $\mu > 1$. Then $\lim_{n\to\infty} \mathrm{E}(Z_n) = \infty$.
- We can prove that if $\lim_{n\to\infty} \mathrm{E}(Z_n) = 0$ then the probability of extinction is 1.

Variance of the generation size

- To compute the variance, we may use the law of total variance:

$$\operatorname{Var}(Z_n) = \operatorname{Var}(\operatorname{E}(Z_n \mid Z_{n-1})) + \operatorname{E}(\operatorname{Var}(Z_n \mid Z_{n-1}))$$

- Using the notation $\mu = \operatorname{E}(X_i)$ and $\sigma^2 = \operatorname{Var}(X_i)$, we get

$$\begin{aligned}
\operatorname{Var}(Z_n) &= \operatorname{Var}(\operatorname{E}(Z_n \mid Z_{n-1})) + \operatorname{E}(\operatorname{Var}(Z_n \mid Z_{n-1})) \\
&= \operatorname{Var}(\mu Z_{n-1}) + \operatorname{E}\left(\sigma^2 Z_{n-1}\right) \\
&= \mu^2 \operatorname{Var}(Z_{n-1}) + \sigma^2 \mu^{n-1}
\end{aligned}$$

- From this we prove by induction

$$\operatorname{Var}(Z_n) = \sigma^2 \mu^{n-1} \sum_{k=0}^{n-1} \mu^k = \begin{cases} n\sigma^2 & \text{if } \mu = 1 \\ \sigma^2 \mu^{n-1}(\mu^n - 1)/(\mu - 1) & \text{if } \mu \neq 1 \end{cases}$$

MVE550 2020 Lecture 6.2
Dobrow Sections 4.3, 4.4
Probability generating functions
Extinction of Branching processes

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Probability generating functions

- For *any* discrete random variable X taking values in $\{0, 1, 2, \ldots, \}$ define the probability generating function $G(s)$, or $G_X(s)$, as

$$G(s) = \mathrm{E}\left(s^X\right) = \sum_{k=0}^{\infty} s^k \Pr\left(X = k\right).$$

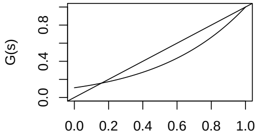
- The series converges absolutely for $|s| \leq 1$.
- We get a 1-1 correspondence between probability vectors on $\{0, 1, 2, \ldots, \}$ and functions represented by a series in this way.
- If $G_X(s) = G_Y(s)$ for all s for random variables X and Y then X and Y have the same distribution.
- The correspondence of X with $G_X(s)$ provides an important and surprisingly useful computational tool.

What does $G_X(s)$ look like?

- $G_X(1) = 1$ and $G_X(0) = \Pr(X = 0)$.
- We get

$$
\begin{aligned}
G'(s) &= \sum_{k=1}^{\infty} ks^{k-1} \Pr(X = k) = \mathrm{E}\left(Xs^{X-1}\right) \\
G''(s) &= \sum_{k=2}^{\infty} k(k-1)s^{k-2} \Pr(X = k) = \mathrm{E}\left(X(X-1)s^{X-2}\right) \\
G'''(s) &= \sum_{k=3}^{\infty} k(k-1)(k-2)s^{k-3} \Pr(X = k) = \mathrm{E}\left(X(X-1)(X-2)s^{X-3}\right)
\end{aligned}
$$

- As a consequence, $G'(s)$ and $G''(s)$ are positive $s \in (0, 1)$.
- Example below: $G_X(s)$ when $X \sim \text{Binomial}(10, 0.2)$. (The diagonal is added to the plot).



Some properties of probability generating functions

- To go from X to $G_X(s)$: Compute the infinite sum.
- To go from $G_X(s)$ to X: Use that we have

$$P(X = j) = \frac{G^{(j)}(0)}{j!}.$$

- If X and Y are independent,

$$G_{X+Y}(s) = \mathrm{E}\left(s^{X+Y}\right) = \mathrm{E}\left(s^X s^Y\right) = \mathrm{E}\left(s^X\right)\mathrm{E}\left(s^Y\right) = G_X(s)G_Y(s)$$

- $\mathrm{E}\left(X\right) = G'(1)$
- $\mathrm{E}\left(X(X-1)\right) = G''(1)$.
- As a consequence, $\mathrm{Var}\left(X\right) = G''(1) + G'(1) - G'(1)^2$.

Probability generating functions for Branching processes

Assume we have a Branching process $Z_0, Z_1, \dots,$ with random variables X_k counting the offspring at each node.

- Write $G_n(s) = G_{Z_n}(s) = \mathrm{E}\left(s^{Z_n}\right)$ and $G(s) = G_{X_k}(s) = \mathrm{E}\left(s^{X_k}\right)$.
- We get

$$\begin{aligned} G_n(s) &= \mathrm{E}\left(s^{\sum_{k=1}^{Z_{n-1}} X_k}\right) = \mathrm{E}\left(\mathrm{E}\left(s^{\sum_{k=1}^{Z_{n-1}} X_k} \mid Z_{n-1}\right)\right) \\ &= \mathrm{E}\left(\mathrm{E}\left(\prod_{k=1}^{Z_{n-1}} s^{X_k} \mid Z_{n-1}\right)\right) = \mathrm{E}\left(G(s)^{Z_{n-1}}\right) = G_{n-1}(G(s)). \end{aligned}$$

- As $G_0(s) = \mathrm{E}\left(s^{Z_0}\right) = s$, it follows that $G_n(s) = G(G(G(\dots G(s)\dots)))$, with n iterations of the G function.
- This result can be applied numerically to compute $G_n(s)$, but it is even more important theoretically.

Extinction probability theorem

THEOREM

- Let G be the probability generating function for the offspring distribution for a branching process. The probability of eventual extinction is the smallest positive root of the equation $s = G(s)$.
- In the (subcritical and) critical cases, the extinction probability is 1.

- Proof: Let e_n be the probability that the process is extinct in generation n. Then

  $$e_n = \Pr(Z_n = 0) = G_n(0) = G(G_{n-1}(0)) = G(\Pr(Z_{n-1} = 0)) = G(e_{n-1})$$

  We get for the probability of extinction

  $$e = \lim_{n\to\infty} e_n = \lim_{n\to\infty} G(e_{n-1}) = G(\lim_{n\to\infty} e_{n-1}) = G(e)$$

  so e is a root of G. Starting with any root $0 < x$ and applying the increasing function G repeatedly on both sides yields $e \leq x$.