MVE550 2020 Lecture 11.1
Dobrow Sections 7.4, 7.5
Absorbing states. Time Reversibility.

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Review / motivation

- We are studying the properties of continuous time Markov chains with discrete state spaces.
- For an irreducible chain with finite state space there is a unique stationary distribution v which is also the limiting distribution, satisfying $vQ = 0$, where Q is the generator matrix of the process.
- We now turn to questions such as: What is the expected time until we get to some state? This can be answered by making the state into an *absorbing state* and computing the expected time to absorbtion.
- We also look at global balance, local balance, and time reversibility.

Absorbing states

- Assume $\{X_t\}_{t\geq 0}$ is a continuous-time Markov chain with k states. Assume the last state is absorbing and the rest are not. (They are then transient).
- We have that $q_k = 0$ and the entire last row must consist of zeros. We get

$$Q = \begin{bmatrix} V & * \\ \mathbf{0} & 0 \end{bmatrix}.$$

- Let F be the $(k-1) \times (k-1)$ matrix so that F_{ij} is the expected time spent in state j when the chain starts in i. We can shown that $F = -V^{-1}$ (see next page).
- Note that, if the chain starts in state i, the expected time until absorbtion is the sum of the i'th row of F. Thus the expected times until absorbtion are given by the matrix product $F1$ of F with a column of 1's.

Outline of proof (different from Dobrow's)

- Generally, define D as the matrix with $(1/q_1, \dots, 1/q_k)$ along its diagonal, with all other entries zero. If there are no absorbing states

$$\tilde{P} = DQ + I$$

- Write A_- for a square matrix without its last row and column.
- If the last state is absorbing, so that $q_k = 0$, we get

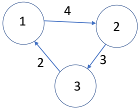
$$\tilde{P}_- = D_- Q_- + I$$

- Let F' be the matrix where F'_{ij} is the expected *number of stays* in state j before absorbtion when starting in state i. As the lengths of stays and changes in states are independent, we get $F = F' D_-$.
- From the theory of Chapter 3, we have that $F' = (I - \tilde{P}_-)^{-1}$.
- We get

$$F = F' D_- = (I - \tilde{P}_-)^{-1} D_- = (-D_- Q_-)^{-1} D_- = (-Q_-)^{-1}.$$

Global Balance

- If $v = (v_1, v_2, v_3)$ the stationary distribution of chain below, the flow into



  a state must be equal to the flow out of that state.
- We get equations $4v_1 = 2v_3$, $3v_2 = 4v_1$, and $2v_3 = 3v_2$.
- Note that these are exactly the equations we get from $vQ = 0$:

$$(v_1, v_2, v_3) \begin{bmatrix} -4 & 4 & 0 \\ 0 & -3 & 3 \\ 2 & 0 & -2 \end{bmatrix} = 0$$

- This happens because $vQ = 0$ gives for each state j

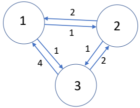
$$\sum_{i \neq j} v_i q_{ij} = v_j q_j$$

- These are called the global balance equations.
- Generalization: If A is a set of states, then the long term rates of movement *into* and *out of* A are the same:

$$\sum_{i \in A} \sum_{j \notin A} v_i q_{ij} = \sum_{i \in A} \sum_{j \notin A} v_j q_{ji}$$

Local balance and time reversibility

- A stronger condition: The flow between *every pair* of states is balanced. This is *not* true for all models!
- From the model below we now get the equations



$$1v_1 = 2v_2,\ 1v_2 = 2v_3, \text{ and } 4v_3 = 1v_1.$$

- A continuous-time Markov chain with unique stationary distribution v is said to be *time reversible* if for all i, j,

$$v_i q_{ij} = v_j q_{ji}$$

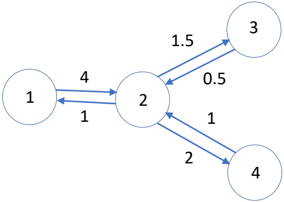
- This is called the *local balance* condition.
- Note: The rate of observed changes from i to j is the same as the rate of observed changes from j to i. Thus this is also called *time reversibility*.
- Note that (similar to discrete chains): If a probability vector v satisfies local balance condition, then v is the unique stationary distribution. (Easy to show).

Markov processes with transition graphs that are trees

- A *tree* is a graph that does not contain cycles.
- Assume the transition graph of an irreducible continuous-time Markov chain is a tree.
- In a tree, any edge between two states divides all states into two groups (one on each side of the edge). Thus, the flow must be balanced across each edge.
- It follows that the Markov chain must satisfy the local balance condition, i.e., be time reversible, i.e., $v_i q_{ij} = v_j q_{ji}$ for all i and j.
- More formally, this can be proved using the generalized global balance property.
- Note that the process can be time reversible even if the transition graph is not a tree.

Example

- Consider the continuous-time Markov chain with transition graph



- As the transition graph is a tree, the chain is necessarily time reversible. We can find the stationary distribution by considering the local balance equations:

$$4v_1 = 1v_2, \qquad 1.5v_2 = 0.5v_3, \qquad 2v_2 = 1v_4$$

- Together with the equation $v_1 + v_2 + v_3 + v_4 = 1$ we easily get the limiting distribution

$$v = \left(\frac{4}{25}, \frac{1}{25}, \frac{12}{25}, \frac{8}{25} \right)$$

Birth-and-death processes

- A birth-and-death process is a continuous-time Markov chain where the state space is the set of nonnegative integers and transitions only occur to neighbouring integers.
- The process is necessarily time-reversible, as the transition graph is a tree (in fact, a line).
- We denote the rate of *births* from i to $i+1$ with λ_i, and the rate of *deaths* from i to $i-1$ with μ_i.
- The generator matrix is

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \dots \\ \mu_1 & -(\mu_1+\lambda_1) & \lambda_1 & 0 & \dots \\ 0 & \mu_2 & -(\mu_2+\lambda_2) & \lambda_2 & \dots \\ 0 & 0 & \mu_3 & -(\mu_3+\lambda_3) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- Provided $\sum_{k=0}^{\infty} \prod_{k=1}^{\infty} \frac{\lambda_{i-1}}{\mu_i} \leq \infty$, the unique stationary distribution is given by

$$v_k = v_0 \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i} \text{for } k = 1, 2, \dots, \qquad v_0 = \left(\sum_{k=0}^{\infty} \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i} \right)^{-1}$$

MVE550 2020 Lecture 11.2
Dobrow Sections 7.6, 7.7
Queueing theory. Poisson subordination.

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Review / overview

- We are still working with continuous time discrete state space Markov chains.
- We discussed a limiting theorem, and that Markov chains where the transition graph is a tree are time reversible: One can then find the limiting distribution by solving the local balance equations.
- We review the definition of birth-and-death processes and look at an example.
- We look at some queueing theory.
- We discuss Poisson subordination.

Birth-and-death processes

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Example

- The simplest example of a birth-and-death process is one where all birth rates λ_i and all death rates μ_i are the same values λ and μ, respectively.
- We get that

$$
\begin{aligned}
v_k &= v_0 \prod_{i=1}^{k} \frac{\lambda}{\mu} = v_0 \left(\frac{\lambda}{\mu}\right)^k \\
v_0 &= \left(\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k\right)^{-1} = \frac{1}{1 + \frac{\lambda}{\mu} + (\frac{\lambda}{\mu})^2 + \dots} = \frac{1}{1/(1 - \frac{\lambda}{\mu})} = 1 - \frac{\lambda}{\mu}
\end{aligned}
$$

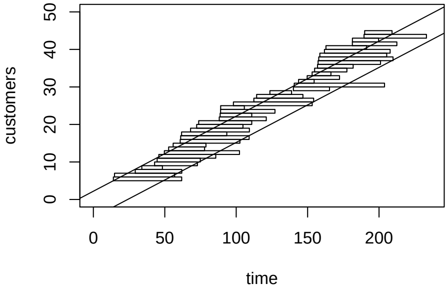
- We see that the limiting distribution is Geometric $\left(1 - \frac{\lambda}{\mu}\right)$.
- For example, the long-term average value of X_t will be

$$
\frac{\lambda/\mu}{1 - \lambda/\mu} = \frac{\lambda}{\mu - \lambda}
$$

Queueing theory

- Birth-and-death processes are special cases of queues.
- In the more general theory of queues:
  - The arrival process ("births") need not be a Poisson process, with exponentially distributed inter-arrival times.
  - The service times in the system need not be exponentially distributed.
  - There can be many other generalizations, such as how many servers there are, how the servers work, how the line works, etc.
- One can use notation $A/B/n$ where A denotes arrival process, B denotes service time distribution, and n the number of servers.
- With this notation, our birth-and-death model above with constant birth and death rates is denoted $M/M/1$. (M is Markov).
- Our formulas above also give the limiting distribution for an $M/M/c$ queue, where there are c different servers.

Little's formula



The boxes represent customers arriving at a rate λ and staying for an average time W. The left line represents the average arrival times of customers: It has slope λ. The right line represents the average departure time of customers. The horizontal distance between the lines is W. The vertical distance between the lines will be L, the average number of customers in the system. Thus

$$\lambda = \frac{L}{W}$$

Poisson subordination

- Instead of simulating from a continuous time finite state Markov chain by drawing the holding time from Exponential(q_i), where q_i depends on the state i, simulate a holding time from Exponential(λ) where λ is large, and allow movement back to the same state.
- Matematical formulation: Given generator matrix Q. If $\lambda \geq \max(q_, \ldots, q_k)$ then
  - $R = \frac{1}{\lambda}Q + I$ is a stochastic matrix.
  - We can write

$$P(t) = e^{tQ} = e^{-t\lambda I} e^{t\lambda R} = e^{-t\lambda} \sum_{k=0}^{\infty} \frac{(t\lambda R)^k}{k!} = \sum_{k=0}^{\infty} R^k \frac{e^{-\lambda t}(\lambda t)^k}{k!}.$$

- Thus: To find the probability of going from i to j during time t:
  1. Simulate the number of changes occurring $k \sim \text{Poisson}(\lambda t)$.
  2. Move the discrete chain with transition matrix R k steps.
- This provides a good way to compute e^{tQ}: Throw away terms where k is over some limit. Better accuracy that using definition of exponential matrix!
- Discrete and continuous chain have same stationary distributions!