Basic Stochastic Processes: Financial applications

Lecture 1 (26 November 2020)

Finite probability theory

Let $\Omega = \{\omega_1, \ldots, \omega_m\}$ be a sample space containing m elements.

Let $p = (p_1, \ldots, p_m)$ be a **probability vector**, i.e.,

$$0 < p_i < 1, \text{ for all } i = 1, \ldots, m, \quad \text{and} \quad \sum_{i=1}^{m} p_i = 1.$$

We define $p_i = \mathbb{P}(\{\omega_i\})$ to be the probability of the event $\{\omega_i\}$.

If $A \subseteq \Omega$ is a non-empty event, we define the probability of A as

$$\mathbb{P}(A) = \sum_{i:\omega_i \in A} p_i = \sum_{\omega \in A} \mathbb{P}(\{\omega\}).$$

Moreover $\mathbb{P}(\emptyset) = 0$. The pair $(\Omega, \mathbb{P})$ is called a **finite probability space**.

Example

For example, given $p \in (0,1)$, the probability space

$$\Omega_N = \{H,T\}^N, \quad \mathbb{P}_p(\{\omega\}) = p^{N_H(\omega)}(1-p)^{N_T(\omega)}$$

is called the N**-coin toss probability space**. Here $N_H(\omega)$ is the number of Heads in the toss $\omega \in \Omega_N$ and $N_T(\omega) = N - N_H(\omega)$ is the number of Tails. In this probability space, tosses are independent and each toss has the same probability p to result in a head.

A **random variable** is a function $X : \Omega \to \mathbb{R}$. Y is said to be X**-measurable** if there exists a function g such that $Y = g(X)$.

Two random variables X, Y are **independent** if $\mathbb{P}(X \in I, Y \in J) = \mathbb{P}(X \in I)\mathbb{P}(Y \in J)$ for every $I \subseteq \mathrm{Im}(X)$ and $J \subseteq \mathrm{Im}(Y)$, where $\mathrm{Im}(X) = \{y \in \mathbb{R} : y = X(\omega) \text{ for some } \omega \in \Omega\}$ is the image of X.

The function

$$f_X(x) = \mathbb{P}(X = x),$$

is called the **probability density function** (or probability mass function) of X. Clearly $f_X(x) = 0$ if $x \notin \mathrm{Im}(X)$.

The **expectation** of X is denoted by $\mathbb{E}[X]$; it is given by

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega)\mathbb{P}(\{\omega\}) = \sum_{x \in \mathrm{Im}(X)} x f_X(x)$$

Theorem 0.1 Let X, Y be random variables, $g : \mathbb{R} \to \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$. The following holds:

1. $\mathbb{E}[\alpha X + \beta Y] = \alpha\mathbb{E}[X] + \beta\mathbb{E}[Y]$ (linearity).
2. If $X \geq 0$ and $\mathbb{E}[X] = 0$, then $X = 0$.
3. If X, Y are independent then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.
4. If $Y = g(X)$, i.e., if Y is X-measurable, then
$$\mathbb{E}[g(X)] = \sum_{x \in \mathrm{Im}(X)} g(x) f_X(x). \tag{1}$$

The quantity

$$\mathrm{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

is called **variance** of the random variable X. The quantity

$$\mathrm{Cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y]) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

is called **covariance** of the random variables X, Y.

If $\mathrm{Var}[X], \mathrm{Var}[Y]$ are both positive (i.e., if X, Y are not deterministic constants), the quantity

$$\mathrm{Corr}[X, Y] = \frac{\mathrm{Cov[X, Y]}}{\sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]}} \in [-1, 1]$$

is called **correlation** of X, Y. If $\mathrm{Corr}[X, Y] = 0$, the random variables X, Y are said to be **uncorrelated**.

It follows by Theorem 0.1(3) that X, Y independent $\Rightarrow$ X, Y uncorrelated (while the opposite is in general not true).

The **conditional expectation** of X given Y is denoted by $\mathbb{E}[X|Y]$:

$$\mathbb{E}[X|Y](\omega) = \sum_{x \in \mathrm{Im}(X)} \mathbb{P}(X = x | Y = Y(\omega))x,$$

where $\mathbb{P}(A|B) = \mathbb{P}(B)^{-1}\mathbb{P}(A \cap B)$ is the conditional probability of the event A given the event B.

The conditional expectation is a Y-measurable random variable and satisfies the following properties.

Theorem 0.2

Let $X, Y, Z : \Omega \to \mathbb{R}$ be random variables and $\alpha, \beta \in \mathbb{R}$. Then

1. $\mathbb{E}[\alpha X + \beta Y | Z] = \alpha \mathbb{E}[X|Z] + \beta \mathbb{E}[Y|Z]$ (linearity).
2. If X is independent of Y, then $\mathbb{E}[X|Y] = \mathbb{E}[X]$.
3. If X is Y-measurable, then $\mathbb{E}[X|Y] = X$.
4. $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$.
5. If X is Z-measurable, then $\mathbb{E}[XY|Z] = X\mathbb{E}[Y|Z]$.
6. If Z is Y-measurable then $\mathbb{E}[\mathbb{E}[X|Y]|Z] = \mathbb{E}[X|Z]$.

These properties remain true if the conditional expectation is taken with respect to several random variables.

A **discrete stochastic process** is a (possibly finite) sequence $\{X_0, X_1, \dots\} = \{X_n\}_{n\in\mathbb{N}}$ of random variables. We refer to the index n in X_n as **time step**.

If the discrete stochastic process is finite, i.e., if it runs only for a finite number $N \geq 1$ of time steps, we shall denote it by $\{X_n\}_{n=0,\dots,N}$ and call it a **N-period process**.

At each time step, a discrete stochastic process on a finite probability space is a random variable with finitely many possible values. More precisely, for all $n = 0, 1, 2, \dots,$ the value x_n of X_n satisfies $x_n \in \mathrm{Im}(X_n)$. We call x_n an **admissible state** of the stochastic process. Note that x_n is an admissible state if and only if $\mathbb{P}(X_n = x_n) > 0$.

A stochastic process $\{Y_n\}_{n\in\mathbb{N}}$ is said to be **measurable** with respect to $\{X_n\}_{n\in\mathbb{N}}$ if for all $n \in \mathbb{N}$ there exists a function $g_n : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $Y_n = g_n(X_0, X_2, \dots, X_n)$.

If $Y_n = h_n(X_0, \dots, X_{n-1})$ for some function $h_n : \mathbb{R}^n \to \mathbb{R}$, $n \geq 1$, then $\{Y_n\}_{n\in\mathbb{N}}$ is said to be **predictable** from the process $\{X_n\}_{n\in\mathbb{N}}$.

A discrete stochastic process $\{X_n\}_{n\in\mathbb{N}}$ on the finite probability space $(\Omega, \mathbb{P})$ is called a **martingale** if

$$\mathbb{E}[X_{n+1}|X_1, X_2, \dots X_n] = X_n, \quad \text{for all } n \in \mathbb{N}.$$

Martingales have constant expectation, i.e., $\mathbb{E}[X_n] = \mathbb{E}[X_0]$, for all $n \in \mathbb{N}$.

A discrete stochastic process $\{X_n\}_{n\in\mathbb{N}}$ on the finite probability space $(\Omega, \mathbb{P})$ is called a **Markov chain** if it satisfies the **Markov property**:

$$\mathbb{P}(X_{n+1} = x_{n+1}|X_n = x_n) = \mathbb{P}(X_{n+1} = x_{n+1}|X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n),$$

for all $n \in \mathbb{N}$ and for all admissible states $x_0 \in \mathrm{Im}(X_0), \ldots, x_{n+1} \in \mathrm{Im}(X_{n+1})$ such that $\mathbb{P}(X_0 = x_0, X_1 = x_1, \ldots X_n = x_n)$ is positive.

The left hand side is called the **transition probability** from the state x_n to the state x_{n+1} and is denoted also as $\mathbb{P}(x_n \to x_{n+1})$.

If $\mathbb{P}(x_n \to x_{n+1})$ is independent of $n = 1, 2, \ldots,$ the Markov process is said to be **time homogeneous**.

Important remark:

The Markov property and the martingale property depend on the probability measure, i.e., a stochastic process can be a martingale and/or a Markov process in one probability $\mathbb{P}$ and neither of them in another probability $\widetilde{\mathbb{P}}$.

Example: Random Walk.

Consider the following stochastic process $\{X_n\}_{n=1,\dots,N}$ defined on the N-coin toss probability space $(\Omega_N, \mathbb{P}_p)$:

$$\omega = (\gamma_1, \dots, \gamma_N) \in \Omega_N, \quad X_n(\omega) = \begin{cases} 1 & \text{if } \gamma_n = H \\ -1 & \text{if } \gamma_n = T \end{cases}.$$

The random variables $X_1, \dots, X_N$ are independent and identically distributed (**i.i.d**), namely

$$\mathbb{P}_p(X_n = 1) = p, \quad \mathbb{P}_p(X_n = -1) = 1 - p, \quad \text{for all } n = 1, \dots, N.$$

Hence

$$\mathbb{E}[X_n] = 2p - 1, \quad \mathrm{Var}[X_n] = 4p(1-p), \quad \text{for all } n = 1, \dots, N.$$

Now, for $n = 1, \dots, N$, let

$$M_0 = 0, \quad M_n = \sum_{i=1}^{n} X_i.$$

The stochastic process $\{M_n\}_{n=0,\dots,N}$ is measurable (but not predictable) with respect to the process $\{X_n\}_{n=1,\dots,N}$ and is called **(N-period) random walk**. It satisfies

$$\mathbb{E}[M_n] = n(2p-1), \quad \text{for all } n = 0, \dots, N.$$

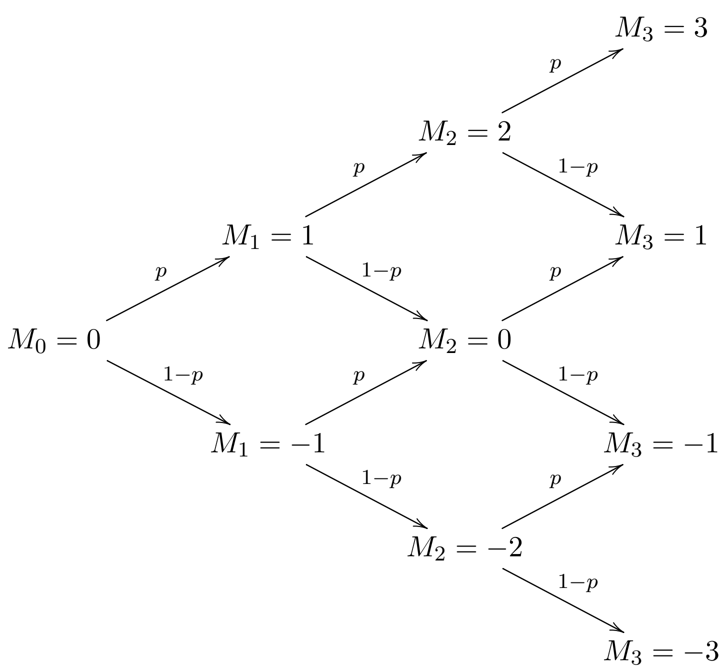
Moreover, since it is the sum of independent random variables, the random walk has variance given by

$$\mathrm{Var}[M_0] = 0, \quad \mathrm{Var}[M_n] = \mathrm{Var}(X_1 + X_2 + \cdots + X_n) = \sum_{i=1}^{n} \mathrm{Var}[X_i] = 4np(1-p).$$

When $p = 1/2$, the random walk is said to be **symmetric**. In this case $\{M_n\}_{n=0,\ldots,N}$ satisfies $\mathbb{E}[M_n] = 0$ and $\mathrm{Var}[M_n] = n$, $n = 0, \ldots, N$.

When $p \neq 1/2$, $\{M_n\}_{n=0,\ldots,N}$ is called an **asymmetric** random walk, or a random walk with **drift**.

If $M_n = k$ then M_{n+1} is either $k+1$ (with probability p), or $k-1$ (with probability $1-p$). Hence we can represent the paths of the random walk by using a binomial tree, as in the following example for $N = 3$:



By inspection we see that the admissible states of the symmetric random walk at the step n are given by

$$\mathrm{Im}(M_n) = \{-n, -n+2, -n+4, \dots, n-2, n\} = \{2k-n,\ k=0,\dots,n\},$$

where k is the number of times that the random walk "goes up" up to the step n included.

The density of M_n is given by the **binomial probability density** function

$$f_{M_n}(x) = \binom{n}{k} p^k (1-p)^{n-k} \delta(x-(2k-n)),\ k=0,\dots,n,$$

where $\delta(z)=1$ if $z=0$ and $\delta(z)=0$ otherwise.

Let $m_0=0$, $m_1 \in \{-1,1\} = \mathrm{Im}(M_1)$, $\dots$, $m_N \in \{-N, -N+2, \dots, N-2, N\} = \mathrm{Im}(M_N)$ be the admissible states at each time step. From the binomial tree of the process it is clear that there exists a path connecting $m_0, m_1, \dots, m_N$ if and only if $m_n = m_{n-1} \pm 1$, for all $n=1,\dots,N$, and we have

$$\begin{aligned}\mathbb{P}(M_n = m_n | M_1 = m_1, \dots, M_{n-1} = m_{n-1}) &= \mathbb{P}(M_n = m_n | M_{n-1} = m_{n-1}) \\ &= \begin{cases} p & \text{if } m_n = m_{n-1}+1 \\ 1-p & \text{if } m_n = m_{n-1}-1 \end{cases}.\end{aligned}$$

Hence the random walk is an example of time homogeneous Markov chain.

Next we show that the *symmetric* random walk is a martingale.

Using the linearity of the conditional expectation we have

$$\begin{aligned}\mathbb{E}[M_n|M_1,\dots,M_{n-1}] &= \mathbb{E}[M_{n-1}+X_n|M_1,\dots,M_{n-1}]\\ &= \mathbb{E}[M_{n-1}|M_1,\dots,M_{n-1}]+\mathbb{E}[X_n|M_1,\dots,M_{n-1}].\end{aligned}$$

As M_{n-1} is measurable with respect to $M_1,\dots,M_{n-1}$, then

$$\mathbb{E}[M_{n-1}|M_1,\dots,M_{n-1}] = M_{n-1},$$

see Theorem 0.2(3).

As X_n is independent of $M_1,\dots,M_{n-1}$, Theorem 0.2(2) gives

$$\mathbb{E}[X_n|M_1,\dots,M_{n-1}] = \mathbb{E}[X_n] = 0.$$

It follows that $\mathbb{E}[M_n|M_1,\dots,M_{n-1}] = M_{n-1}$, i.e., the symmetric random walk is a martingale.

However the asymmetric random walk ($p \neq 1/2$) is *not* a martingale, as it follows by the fact that its expectation $\mathbb{E}[M_n] = n(2p-1)$ is not constant.

Generalized random walk.

A random walk may be defined as any discrete stochastic process $\{M_n\}_{n\in\mathbb{N}}$ which satisfies the following properties:

- $\text{Im}(M_n) = \{-n, -n+2, -n+4, \dots, n-2, n\}$, for all $n = 0, 1, \dots$;
- $\{M_n\}_{n\in\mathbb{N}}$ is a time-homogeneous Markov chain;
- There exists $p \in (0,1)$ such that for $(m_{n-1}, m_n) \in \text{Im}(M_{n-1}) \times \text{Im}(M_n)$, the transition probability $\mathbb{P}(m_{n-1} \to m_n)$ is given by

$$\mathbb{P}(m_{n-1} \to m_n) = \begin{cases} p & \text{if } m_n = m_{n-1} + 1 \\ 1-p & \text{if } m_n = m_{n-1} - 1 \\ 0 & \text{otherwise} \end{cases}.$$

We may generalize this definition by relaxing the second and third properties as follows.

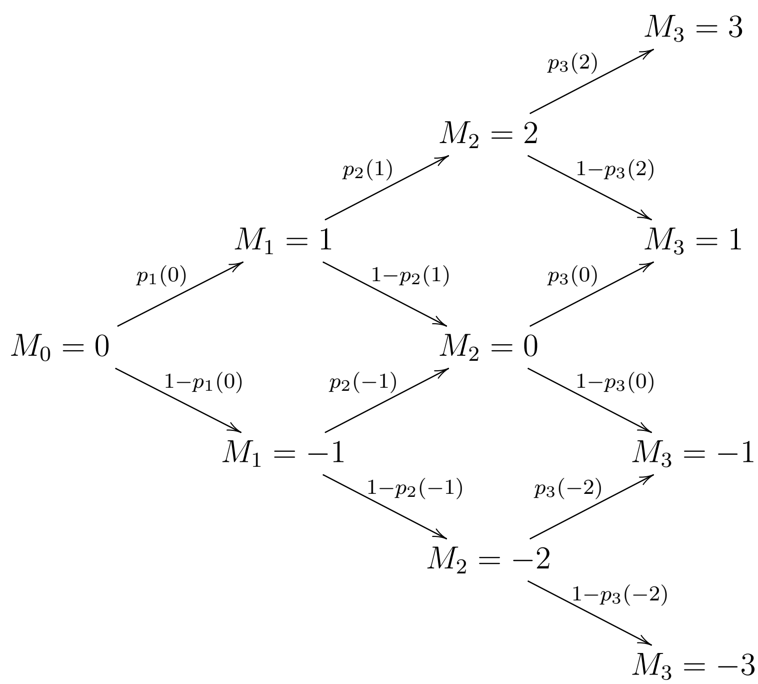
Definition 0.1

A discrete stochastic process $\{M_n\}_{n\in\mathbb{N}}$ on a finite probability space is called a **generalized random walk** if it satisfies the following properties:

1. $\text{Im}(M_n) = \{-n, -n+2, -n+4, \dots, n-2, n\}$, for all $n = 0, 1, \dots$;
2. $\{M_n\}_{n\in\mathbb{N}}$ is a Markov chain;
3. For all $n = 1, 2, \dots$ there exist $p_n : \text{Im}(M_{n-1}) \to (0,1)$ such that

$$\mathbb{P}(m_{n-1} \to m_n) = \begin{cases} p_n(m_{n-1}) & \text{if } m_n = m_{n-1} + 1 \\ 1-p_n(m_{n-1}) & \text{if } m_n = m_{n-1} - 1 \\ 0 & \text{otherwise} \end{cases}.$$

The binomial tree of a generalized random walk will be written as in the following example:



Remark

The admissible states of a generalized random walk are precisely the same as for the standard random walk, but they are now attained with different probabilities. In particular the generalized random walk is no longer binomially distributed, unless of course $p_n \equiv p$ for all $n = 1, 2, \ldots$.

It is clear that any path in the N-period random walk is uniquely identified by a vector $x \in \{-1, 1\}^N$, i.e., a N-dimensional vector where each component is either -1 or 1. More precisely, the path of the random walk corresponding to $x \in \{-1, 1\}^N$ it the unique path satisfying $M_0 = 0$ and $M_i = M_{i-1} + x_i$, $i = 1, \ldots, N$.

Theorem 0.3

Let $x \in \{-1, 1\}^N$ and set $x_0 = 0$. The probability $\mathbb{P}(x)$ that the generalized random walk follows the path x is given by

$$\mathbb{P}(x) = \prod_{t=1}^{N} \left[-\min(x_t, 0) + x_t p_t \left(\sum_{j=0}^{t-1} x_j \right) \right].$$

Example

In the 3-period model consider the path $x = (-1, -1, 1)$. Then according to the previous theorem

$$\begin{aligned}\mathbb{P}((-1,-1,1)) &= (-\min(-1,0) + (-1)p_1(0))(-\min(-1,0) + (-1)p_2(0-1)) \\ &\quad \times (-\min(1,0) + (1)p_2(0-1-1)) = (1-p_1(0))(1-p_2(-1))p_2(-2).\end{aligned}$$

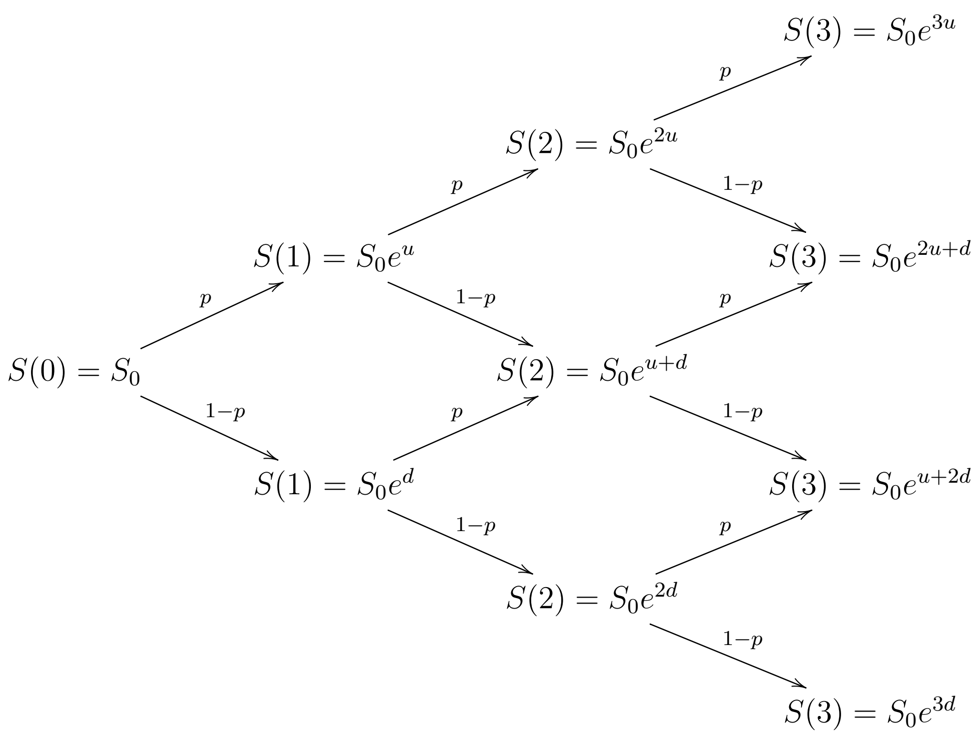
That this formula is correct is easily seen in the binomial tree above.

Review of the binomial model with constant risk-free rate

Given $0 < p < 1$, $S_0 > 0$ and $u > d$, the **binomial stock price** at time t is given by $S(0) = S_0$ and

$$S(t) = \begin{cases} S(t-1)e^{u} & \text{with probability } p \\ S(t-1)e^{d} & \text{with probability } 1-p \end{cases}, \qquad \text{for } t = 1, \dots, N.$$

For instance, for $N = 3$ the binomial stock can be represented as in the following recombining binomial tree:



The possible stock prices at time t belong to the set

$$\mathrm{Im}(S(t)) = \{S_0 e^{ku+(t-k)d},\ k(t) = 0,\ldots,t\},$$

where k is the number of times that the price goes up up to and including time t. It follows that there are $t+1$ possible prices at time t and so the number of nodes in the binomial tree grows linearly in time.

Moreover the stock price is binomially distributed, namely

$$f_{S(t)}(x) = \binom{t}{k} p^k (1-p)^{t-k} \delta(x - S_0 e^{ku+(t-k)d}),\ k = 0,\ldots,t.$$

The binomial stock price can be interpreted as a stochastic process defined on the N-coin toss probability space $(\Omega_N, \mathbb{P}_p)$. To see this, consider the following i.i.d. random variables

$$X_t : \Omega_N \to \mathbb{R}, \quad X_t(\omega) = \left\{ \begin{array}{ll} 1, & \text{if the } t^{th} \text{ toss in } \omega \text{ is } H \\ -1, & \text{if the } t^{th} \text{ toss in } \omega \text{ is } T \end{array} \right., \quad t = 1,\ldots,N.$$

We can rewrite the binomial stock price as

$$S(t) = S(t-1)\exp[(u+d)/2 + (u-d)X_t/2]$$

which upon iteration leads to

$$S(t) = S_0 \exp\left[t\left(\frac{u+d}{2}\right) + \left(\frac{u-d}{2}\right)M_t\right], \quad M_t = X_1 + \cdots + X_t,\ t = 1,\ldots,N.$$

Hence $S(t) : \Omega_N \to \mathbb{R}$ and therefore $\{S(t)\}_{t=0,\ldots,N}$ is a N-period stochastic process on the N-coin toss probability space $(\Omega_N, \mathbb{P}_p)$.

In this context, $\mathbb{P}_p$ is called **physical** (or **real-world**) **probability measure**, to distinguish it from the martingale (or risk-neutral) probability introduced below.

Letting $M_0 = 0$, we have that $\{M_t\}_{t=0,\dots,N}$ is a random walk (which is asymmetric for $p \neq 1/2$).

It follows that $\{S(t)\}_{t=0,\dots,N}$ is measurable, but not predictable, with respect to $\{M_t\}_{t=0,\dots,N}$.

For each $\omega \in \Omega_N$, the vector $(S(0), S(1,\omega), \dots, S(N,\omega))$ is called a **path** of the binomial stock price.

Binomial market

A **binomial market** is a market that consists of one stock with price

$$S(0) = S_0 > 0, \quad S(t) = \begin{cases} S(t-1)e^u & \text{with probability } p \\ S(t-1)e^d & \text{with probability } 1-p \end{cases}, \qquad \text{for } t = 1, \dots, N.$$

and a **risk-free asset** with value $B(t)$ at time $t = 1, \dots, N$.

In the standard binomial model it is assumed that $B(t)$ is a deterministic function of time with constant interest rate, namely

$$r = \log B(t+1) - \log B(t), \quad \text{or} \quad R = \frac{B(t+1) - B(t)}{B(t)}.$$

It follows that the value of the risk-free asset at time t can be written in either of the two forms

$$B(t) = B_0 e^{rt}, \;\; \text{or} \;\; B(t) = B_0(1+R)^t, \quad t = 1, \dots, N, \; B_0 = B(0) > 0.$$

Here R is the **discretely compounded risk-free rate** and r is the **continuously compounded risk-free rate**.

The quantity

$$S^*(t) = e^{-rt}S(t), \quad \text{or equivalently} \quad S^*(t) = \frac{S(t)}{(1+R)^t},$$

is called the **discounted price** of the stock (at time $t = 0$).

In the following we denote by $\mathbb{E}_p$ the (possibly conditional) expectation in the probability space $(\Omega_N, \mathbb{P}_p)$.

Theorem 0.4

If $r \notin (d, u)$, there is no probability measure $\mathbb{P}_p$ on the sample space Ω_N such that the discounted stock price process $\{S^*(t)\}_{t=0,\dots,N}$ is a martingale. For $r \in (d, u)$, $\{S^*(t)\}_{t=0,\dots,N}$ is a martingale with respect to the probability measure $\mathbb{P}_p$ if and only if $p = q$, where

$$q = \frac{e^r - e^d}{e^u - e^d}.$$

Due to Theorem 0.4, $\mathbb{P}_q$ is called **martingale probability measure**.

Moreover, since martingales have constant expectation, then

$$\mathbb{E}_q[S(t)] = S_0 e^{rt}.$$

Thus in the martingale probability measure one expects the same return on the stock as on the risk-free asset. For this reason, $\mathbb{P}_q$ is also called **risk-neutral probability**.

Self-financing portfolios

A **portfolio process** in a binomial market is a stochastic process

$$\{(h_S(t), h_B(t))\}_{t=0,\dots,N}$$

such that, for $t = 1, \dots, N$, $(h_S(t), h_B(t))$ corresponds to the portfolio position (number of shares) on the stock and the risk-free asset held in the interval $(t-1, t]$.

A positive number of shares corresponds to a long position on the asset, while a negative number of shares corresponds to a short position.

As portfolio positions held for one instant of time only are meaningless, we use the convention $h_S(0) = h_S(1), h_B(0) = h_B(1)$, that is to say, $h_S(1), h_B(1)$ is the portfolio position in the *closed* interval $[0, 1]$.

We always assume that the portfolio process is predictable from $\{S(t)\}_{t=0,\dots,N}$, i.e., there exists functions $H_t : (0, \infty)^t \to \mathbb{R}^2$ such that

$$(h_S(t), h_B(t)) = H_t(S(0), \dots, S(t-1)).$$

Thus the decision on which position the investor should take in the interval $(t-1, t]$ depends only on the information available at time $t-1$.

The **value** of the portfolio process is the stochastic process $\{V(t)\}_{t=0,\dots,N}$ given by

$$V(t) = h_B(t)B(t) + h_S(t)S(t), \quad t = 0, \dots, N.$$

A portfolio process $\{(h_S(t), h_B(t))\}_{t=0,\dots,N}$ is said to be **self-financing** if

$$\delta V(t) = h_S(t)\delta S(t) + h_B(t)\delta B(t), \quad t = 1, \dots, N,$$

where $\delta f(t) = f(t) - f(t-1)$, while it is said to generate the **cash flow** $C(t-1)$ if

$$\delta V(t) = h_S(t)\delta S(t) + h_B(t)\delta B(t) - C(t-1), \quad t = 1, \dots, N.$$

Note that $C(t) > 0$ corresponds to cash withdrawn from the portfolio at time t while $C(t) < 0$ corresponds to cash added to the portfolio at time t. The self-financing property means that no cash is ever added or withdrawn from the portfolio.

Theorem 0.5

Let $\{(h_S(t), h_B(t))\}_{t=0,\dots,N}$ be a self-financing predictable portfolio process with value $\{V(t)\}_{t=0,\dots,N}$. Then the discounted portfolio value $\{V^*(t)\}_{t=0,\dots,N}$ is a martingale in the risk-neutral probability measure. Moreover the following identity holds:

$$V^*(t) = \mathbb{E}_q[V^*(N)|S(0), \dots, S(t)], \quad t = 0, \dots, N.$$

Arbitrage portfolios

A portfolio process $\{(h_S(t), h_B(t)\}_{t=0,\dots,N}$ invested in the binomial market is called an **arbitrage portfolio process** if it is predictable and if its value $V(t)$ satisfies

1) $V(0) = 0$;
2) $V(N, \omega) \geq 0$, for all $\omega \in \Omega_N$;
3) There exists $\omega_* \in \Omega_N$ such that $V(N, \omega_*) > 0$.

Theorem 0.6

Assume $d < r < u$, i.e., assume the existence of a risk-neutral probability measure for the binomial market. Then the binomial market is free of self-financing arbitrages.

Proof. Assume that $\{h_S(t), h_B(t)\}_{t=0,\dots,N}$ is a self-financing arbitrage portfolio process. Then $V(0) = V^*(0) = 0$ and since martingales have constant expectation then $\mathbb{E}_q[V^*(t)] = 0$, for all $t = 0, 1, \dots, N$. As $V(N) \geq 0$, then $V^*(N) \geq 0$ and Theorem 0.1(2) entails $V^*(N, \omega) = 0$ for any sample $\omega \in \Omega_N$. Hence $V(N, \omega) = 0$, for all $\omega \in \Omega_N$, contradicting the assumption that the portfolio is an arbitrage. $\square$

Important remark: the existence of a risk-neutral probability measure in not only sufficient but also necessary for the absence of self-financing arbitrages in the binomial market. More precisely, if $r \notin (d, u)$ one can construct self-financing arbitrage portfolios in the market. Hence the binomial market is free of self-financing arbitrages if and only if it admits a risk-neutral probability measure. The latter result is valid for any discrete (or even continuum) market model and is known as the **first fundamental theorem of asset pricing**.

Risk neutral pricing formula for European derivatives in the binomial model

Let $Y : \Omega_N \to \mathbb{R}$ be a random variable and consider the European-style derivative with pay-off Y at maturity time $T = N$. This means that the derivative can only be exercised at time $t = N$.

For standard European derivatives Y is a deterministic function of $S(N)$, i.e.,

$$Y = g(S(N))$$

while for non-standard derivatives Y is a deterministic function of $S(0), \ldots, S(N)$, that is

$$Y = g(S(0), S(1), \ldots, S(N))$$

Examples

- The call option with strike K and maturity N is the standard European derivative with pay-off

$$Y = (S(N) - K)_+ = \max(S(N) - K, 0)$$

- The Asian call option with strike K and maturity T is the non-standard European derivative with pay-off

$$Y = \left(\frac{1}{N+1} \left(\sum_{j=0}^{N} S(j) \right) - K \right)_+$$

Let $\Pi_Y(t)$ be the binomial fair price of the derivative a time t.

By definition, $\Pi_Y(t)$ equals the value $V(t)$ of self-financing, hedging portfolios. In particular, $\Pi_Y(t)$ is a random variable and so $\{\Pi_Y(t)\}_{t=0,\ldots,N}$ is a stochastic process.

Using the hedging condition $V(N) = Y$ (which means $V(N,\omega) = Y(\omega)$, for all $\omega \in \Omega_N$)) and Theorem 0.5, we have the following formula for the fair price at time t of the financial derivative:

$$\Pi_Y(t) = e^{-r(N-t)}\mathbb{E}_q[Y|S(0),\ldots,S(t)].$$

which is known as **risk-neutral pricing formula**. It holds not only for the binomial model but for any discrete—or even continuum —pricing model for financial derivatives. It is used for standard as well as non-standard European derivatives.

For $t = 0$ the risk-neutral pricing formula becomes

$$\Pi_Y(0) = e^{-rN}\mathbb{E}_q[Y].$$

Important remark: We may interpret the previous formula as follows: the current (at time $t = 0$) fair value of the derivative is our expectation on the future payment of the derivative (the pay-off) expressed in terms of the future value of money (discounted pay-off $Y^* = e^{-rN}Y$). The expectation has to be taken *with respect to the martingale probability measure*, i.e., ignoring any (subjective or illegal) estimate on future movements of the stock price (except for the loss in value due to the time-devaluation of money).

Example.

Consider a 2-period binomial model with the following parameters

$$e^u = \frac{4}{3}, \quad e^d = \frac{2}{3}, \quad r = 0, \quad p \in (0,1).$$

Assume further that $S_0 = 36$. Consider the European derivative with pay-off

$$Y = (S(2) - 28)_+ - 2(S(2) - 32)_+ + (S(2) - 36)_+$$

and time of maturity $T = 2$. Using the risk-neutral pricing formula, the fair value of the derivative at $t = 0$ is

$$\Pi_Y(0) = e^{-2r}\mathbb{E}_q[Y] = \mathbb{E}_q[(S(2)-28)_+] - 2\mathbb{E}_q[(S(2)-32)_+] + \mathbb{E}_q[(S(2)-36)_+].$$

By the market parameters we find $q = 1/2$. Hence the distribution of $S(2)$ in the risk-neutral probability measure is

$$\mathbb{P}_q(S(2) = s) = \begin{cases} 1/4 & \text{if } s = 16 \text{ of } s = 64 \\ 1/2 & \text{if } s = 32 \\ 0 & \text{otherwise} \end{cases}.$$

It follows that

$$\mathbb{E}_q[(S(2) - 28)_+] = 11, \quad \mathbb{E}_q[(S(2) - 32)_+] = 8, \quad \mathbb{E}_q[(S(2) - 36)_+] = 7,$$

hence $\Pi_Y(0) = 2$.

By definition of expectation in the N-coin toss probability space, the risk-neutral pricing formula at $t = 0$ for the standard European derivative with pay-off $Y = g(S(N))$ and maturity $T = N$ takes the explicit form

$$\Pi_Y(0) = e^{-rN} \sum_{k=0}^{N} \binom{N}{k} q^k (1-q)^k g(S_0 e^{ku+(N-k)d}).$$

However this formula is not very convenient for numerical computations, because the binomial coefficient $\binom{N}{k}$ will reach very large values for even a relative small number of steps (e.g., $\binom{50}{25}$ is of order 10^{14}). A much more convenient way to compute numerically the binomial price of standard European derivatives is by using the recurrence formula $\Pi_Y(N) = Y$ and

$$\Pi_Y(t) = e^{-r}(q\Pi_Y^u(t+1) + (1-q)\Pi_Y^d(t+1)), \quad t = 0, \dots, N-1,$$

where $\Pi_Y^u(t)$ is the binomial price of the derivative at time t assuming that the stock price goes up at time t, i.e.,

$$\Pi_Y^u(t) = e^{-r(N-t)} \mathbb{E}_q[Y | S(0), \dots, S(t-1), S(t) = S(t-1)e^u]$$

and similarly one defines $\Pi_Y^d(t)$ by replacing "up" with "down". The recurrence formula above follows immediately by the risk-neutral pricing formula and the definition of conditional expectation.

Important Remark: It can be shown that any European derivative in the binomial market can be hedged by a self-financing portfolio invested in the underlying stock and the risk-free asset. For this reason the binomial market is called a **complete market**. In fact, the **second fundamental theorem of asset pricing** states that market completeness is equivalent to the uniqueness of the risk-neutral probability measure. An arbitrage free market is said to be **incomplete** if the risk-neutral measure is not unique. When the market is incomplete the price of European derivatives is not uniquely defined and moreover there exist European derivatives which cannot be hedged by self-financing portfolios. An example of incomplete market is the trinomial model discussed in project 2.

Implementation of the binomial model

For real world applications the binomial model must be properly rescaled in time.

Let $T > 0$ be the maturity of a European derivative and consider the uniform partition of the interval $[0, T]$ with size $h > 0$:

$$0 = t_0 < t_1 < \cdots < t_N = T, \quad t_i - t_{i-1} = h, \quad \text{for all } i = 1, \ldots, N.$$

The binomial stock price on the given partition is given by $S(0) = S_0 > 0$ and

$$S(t_i) = \begin{cases} S(t_{i-1})e^u, & \text{with probability } p, \\ S(t_{i-1})e^d, & \text{with probability } 1-p, \end{cases} \qquad i = 1, \ldots, N,$$

while

$$B(t_i) = B_0 e^{rhi}.$$

The **instantaneous mean of log-return** and the **instantaneous variance** of the binomial stock price are defined respectively by

$$\begin{aligned} \alpha &= \frac{1}{h}\mathbb{E}_p[\log S(t_i) - \log S(t_{i-1})] = \frac{1}{h}[pu + (1-p)d], \\ \sigma^2 &= \frac{1}{h}\mathrm{Var}_p[\log S(t_i) - \log S(t_{i-1})] = \frac{(u-d)^2}{h}p(1-p), \end{aligned}$$

while σ itself is called **instantaneous volatility**.

The parameters α, σ are constant in the standard binomial model and are computed using the physical probability (and *not* the risk-neutral probability).

Inverting the equations above we obtain

$$u = \alpha h + \sigma \sqrt{\frac{1-p}{p}} \sqrt{h}, \quad d = \alpha h - \sigma \sqrt{\frac{p}{1-p}} \sqrt{h}.$$

In the applications of the binomial model it is customary to give the parameters α, σ and then compute u, d.

The risk-neutral probability then becomes

$$q = \frac{e^{rh} - e^{\alpha h - \sigma \sqrt{\frac{p}{1-p}} \sqrt{h}}}{e^{\alpha h + \sigma \sqrt{\frac{1-p}{p}} \sqrt{h}} - e^{\alpha h - \sigma \sqrt{\frac{p}{1-p}} \sqrt{h}}}.$$

The binomial model is trustworthy only for h very small compared to T (i.e., $N >> 1$).

The following Matlab code defines a function

$$\texttt{EuroZeroBin}(\texttt{g}, \texttt{T}, \texttt{s}, \texttt{alpha}, \texttt{sigma}, \texttt{r}, \texttt{p}, \texttt{N})$$

that computes the initial price of the standard European derivative with pay-off $Y = g(S(T))$ using the recurrence formula.

The variable s is the initial price S_0 of the stock.

The function also checks that $q \in (0, 1)$, i.e., that the risk-neutral probability is well defined (and thus the market is free of self-financing arbitrages). If not a message appears which asks to increase the number of steps N.

```
function Pzero=EuroZeroBin(g,T,s,alpha,sigma,r,p,N)
h=T/N;
u=alpha*h+sigma*sqrt(h)*sqrt((1-p)/p);
d=alpha*h-sigma*sqrt(h)*sqrt(p/(1-p));
qu=(exp(r*h)-exp(d))/(exp(u)-exp(d));
qd=1-qu;
if (qu<0 || qd<0)
display('Error:  the market is not arbitrage free.  Increase the
value of N');
Pzero=0;
return
end
S=zeros(N+1,1);
P=zeros(N+1);
S=s*exp((N-[0:N])*u+[0:N]*d).';
P(:,N+1)=g(S);
for j=N:-1:1
for i=1:j
P(i,j)=exp(-r*h)*(qu*P(i,j+1)+qd*P(i+1,j+1));
end
end
```

For instance, upon running the command

$$\texttt{Pzero} = \texttt{EuroZeroBin}(@(\texttt{x})\ \texttt{max}(\texttt{x} - 11, 0), 1/3, 10, 0, 0.5, 0.01, 1/2, 10000)$$

we get the output

$$\texttt{Pzero} = 0.7813,$$

which is the (binomial) price at time $t = 0$ of a European call with strike $K = 11$ and maturity $T = 1/3$ years (4 months) on a stock which at $t = 0$ is priced 10 and which has volatility $\sigma = 0.5$ (i.e., 50%) and zero mean of log-return ($\alpha = 0$). The (annual) risk free rate is $r = 0.01$ (i.e., 1%). Moreover $p = 1/2$ and $N = 10000$.

The binomial price of the derivative is very weakly dependent on the parameter $\alpha \in \mathbb{R}$ and $p \in (0, 1)$ (provided N is sufficiently large, say $N \approx 10000$). Hence one normally chooses $\alpha = 0$ and $p = 1/2$ in the implementation of the binomial model.