Basic Stochastic Processes: Financial applications

Lecture 2 (2 December 2020)

1 A binomial model with stochastic risk-free rate

In this lecture we present a generalization of the binomial model in which the risk-free rate is promoted to a stochastic process. Considering markets with random interest rates is important for the valuation of long maturity contracts.

The binomial model with stochastic interest rate presented in this section will be used in the project on forward and futures contracts.

Let $\{M_t\}_{t=0,\dots,N}$ be a N-period *generalized* random walk with transition probabilities

$$\mathbb{P}(m_{t-1} \to m_t) = \begin{cases} p_t(m_{t-1}) & \text{if } m_t = m_{t-1} + 1 \\ 1 - p_t(m_{t-1}) & \text{if } m_t = m_{t-1} - 1 \\ 0 & \text{otherwise} \end{cases} .$$

We consider a binomial market consisting of a stock with price

$$S(t) = S_0 \exp\left[t\left(\frac{u+d}{2}\right) + \left(\frac{u-d}{2}\right) M_t\right], \quad t = 0, 1, 2, \dots N, \quad (*)$$

together with a risk-free asset with value

$$B(t) = B(t-1)(1 + R(t-1)), \quad t = 1, 2, \dots N, \quad (*)$$

where $\{R(t)\}_{t=0,\dots,N-1}$, $R(t) > -1$, is the discretely compounded interest rate process.

By iterating the previous equation it follows that

$$B(t) = B_0 \prod_{k=0}^{t-1} (1 + R(k)),$$

where $B_0 > 0$ is the initial value of the risk-free asset. As $1 + R(t) > 0$, then $B(t) > 0$ for all $t = 0, \dots, N$.

Note carefully that the possible stock prices at time t in this market are the same as in the standard binomial market model, but now they are attained with a different probability which depends on time and on the price of the stock at time $t-1$. In particular, in the present model $S(t)$ is *not* binomially distributed as it is in the standard binomial model, unless of course we choose $p_n \equiv p$ for all $n = 1, 2, \dots$.

In the following we assume that the risk-free process $\{R(t)\}_{t=0,\dots,N-1}$ is measurable with respect to the generalized random walk $\{M_t\}_{t=0,\dots,N}$. In particular the stochastic process $\{M_t\}_{t=0,\dots,N}$ completely defines the state of the binomial market.

The discounted value (at time $t = 0$) of the stock in the market (*) is defined as $S^*(t) = \frac{B_0}{B(t)}S(t)$, that is

$$S^*(t) = D(t)S(t) = \frac{S(t)}{(1 + R(0))(1 + R(1)) \dots (1 + R(t-1))},$$

where

$$D(0) = 1, \quad D(t) = \prod_{k=0}^{t-1}(1 + R(k))^{-1}, \quad t = 1, \dots, N$$

is the **discount process**. The market is arbitrage free if there exist transition probabilities which make the discounted stock price process $\{S^*(t)\}_{t=0,\dots,N}$ a martingale; if this martingale probability is unique, the market is complete. We discuss below one example.

The Ho-Lee model

The literature abounds of stochastic models for the risk-free rate. In this section we shall study the (discrete) **Ho-Lee model**:

$$R(t) = a(t) + b(t)M_t, \qquad \text{where } a(t) \in \mathbb{R} \text{ and } b(t) > 0, \quad t = 0, 1, 2, \dots, N-1.$$

Since the minimum value of M_t is $-t$, then the condition $R(t) > -1$ is satisfied along all paths if and only if

$$a(t) > b(t)t - 1,$$

which will be assumed from now on.

Theorem 0.7

The market (*) with the Ho-Lee interest rate model admits a martingale probability measure if and only if the functions $a(t), b(t)$ are such that

$$e^d < 1 + a(t) - b(t)t, \quad \text{and} \quad 1 + a(t) + b(t)t < e^u,$$

for all $t = 0, 1, \ldots, N-1$.

Moreover, when it exists, the martingale probability measure is unique and it is given by $p_t(k) = q_t(k)$, where

$$q_t(k) = \frac{1 + a(t-1) + b(t-1)k - e^d}{e^u - e^d},$$

where $t = 1, \ldots, N, k \in \mathrm{Im}(M_{t-1}) = \{-t+1, -t+3, \ldots, t-3, t-1\}$.

Thus, under the above conditions on $a(t), b(t)$, this market (*) is complete.

Remarks

- It is clear that the transition probabilities are constant if and only if $b \equiv 0$ and $a(t) = a(0)$, for all $t = 1, \ldots, N-1$, i.e., if and only if the risk-free rate is a deterministic constant, in which case the market (*) reduces to the standard binomial model
- The previous theorem gives a unique martingale probability, it says (of course) nothing about the physical probability. In the applications of the Ho-Lee model it is also assumed that the *physical* transition probabilities are constants and given by $p_t \equiv p = 1/2$; in particular, $\{M_t\}_{t=0,\ldots,N}$ is a standard symmetric random walk in the physical probability. (In the time-continuum Ho-Lee model, which is the one actually used in the applications, $\{M_t\}_{t=0,\ldots,N}$ is replaced by a Brownian motion, which is the time-continuum limit of the standard symmetric random walk, see later)

Example.

Let a_0, b_0 be constants such that $b_0 > 0$. When

$$a(t) = a(0) := a_0, \quad b(t) = \frac{b_0}{t}, \quad t = 1, \ldots, N-1,$$

the conditions on $a(t), b(t)$ become

$$a_0 > b_0 - 1, \quad e^d < 1 + a_0 - b_0, \quad e^u > 1 + a_0 + b_0$$

and the martingale transition probabilities read

$$q_1(0) = \frac{1 + a_0 - e^d}{e^u - e^d}, \quad q_t(k) = \frac{1 + a_0 + \frac{b_0 k}{t-1} - e^d}{e^u - e^d},$$

for $t = 2, \ldots, N$, $k \in \{-t+1, -t+3, \ldots, t-1\}$.

Note that, in the last example, $a_0 = R(0)$ is the initial value of the risk-free rate, while $b_0 = b(1)$ is the volatility of the risk-free rate in the first time period.

European derivatives on the stock

Next we study the problem of pricing European derivatives in the binomial market with stochastic interest rate.

Definition 0.2

Assume that the market (*) is complete (e.g., the risk-free rate is given by the Ho-Lee model and the conditions on $a(t), b(t)$ in Theorem are verified).

Consider a European derivative with maturity $T = N$ and pay-off Y which is measurable with respect to $M_0, \ldots, M_N$ (e.g., $Y = g(S(N))$ for a standard European derivative on the stock).

The risk-neutral price of the derivative is given by

$$\Pi_Y(t) = D(t)^{-1}\widetilde{\mathbb{E}}[D(T)Y|M_0, \ldots, M_t], \quad t = 0, \ldots, T,$$

where $\widetilde{\mathbb{E}}$ denotes the (conditional) expectation in the martingale probability measure. In particular $\Pi_Y(T) = Y$ and

$$\Pi_Y(0) = \widetilde{\mathbb{E}}[D(T)Y] = \widetilde{\mathbb{E}}[\prod_{k=0}^{T-1}(1 + R(k))^{-1}Y].$$

For example, the **zero coupon bond** (ZCB) with **face value** K and maturity T is the European style derivative that promises to pay K at time T. By the previous definition the value of the ZCB at time t is given by

$$B_K(t, T) = KD(t)^{-1}\widetilde{\mathbb{E}}[D(T)|M_0, \ldots, M_t] \quad t = 0, \ldots, T = N.$$

When $K = 1$ we denote $B_K(t, T)$ simply as $B(t, T)$. Clearly, $B_K(t, T) = KB(t, T)$.

Example in the 3-period model with Ho-Lee risk-free rate.

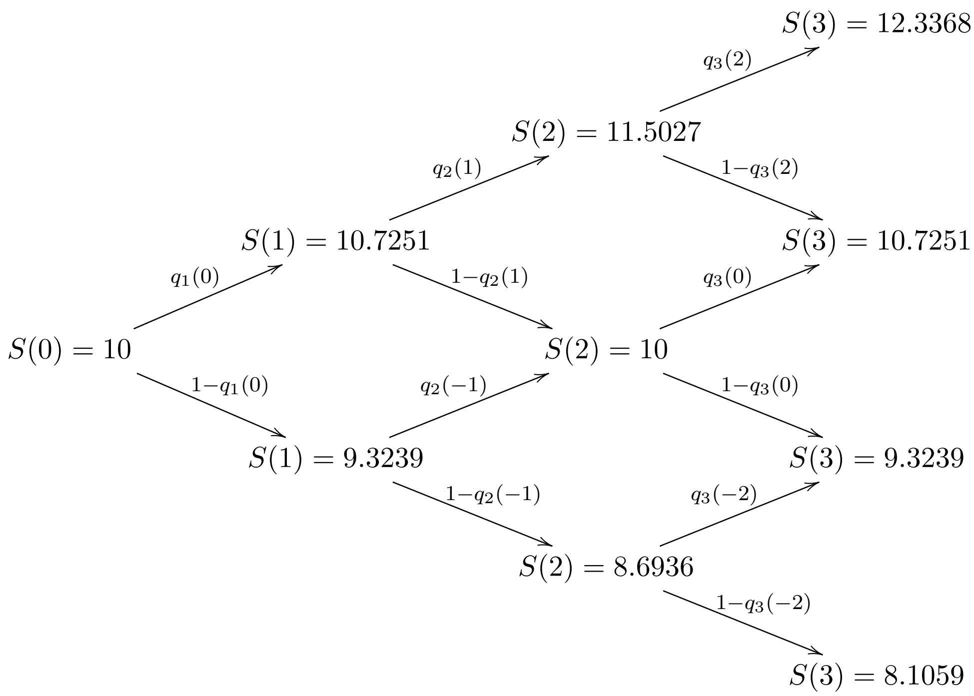
Consider a binomial stock price with $N = 3$, $u = -d = 0.07$, $S_0 = 10$ and a Ho-Lee model for the interest rate with parameters

$$a(0) = R_0 = 0.03, \quad a(1) = 0.05, \quad a(2) = 0.04, \quad b(1) = 0.02, \quad b(2) = 0.01.$$

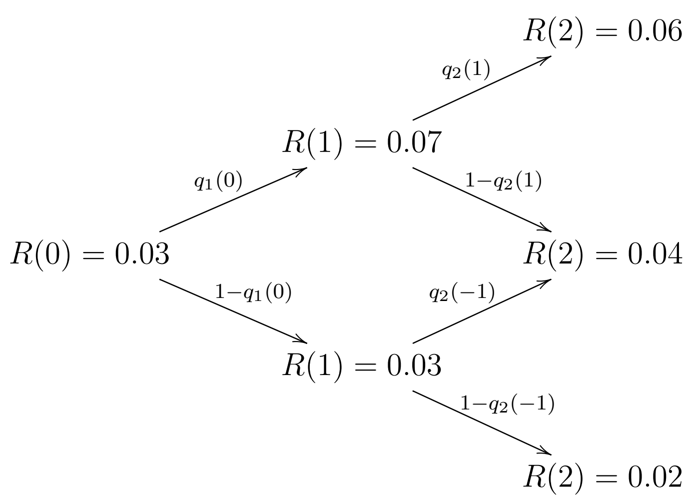
The martingale transition probabilities are

$$\begin{aligned}
q_1(0) &= \frac{1 + R_0 - e^d}{e^u - e^d} = 0.6966 \\
q_2(1) &= \frac{1 + a(1) + b(1) - e^d}{e^u - e^d} = 0.9821 \\
q_2(-1) &= \frac{1 + a(1) - b(1) - e^d}{e^u - e^d} = 0.6966 \\
q_3(2) &= \frac{1 + a(2) + 2b(2) - e^d}{e^u - e^d} = 0.9107 \\
q_3(0) &= \frac{1 + a(2) - e^d}{e^u - e^d} = 0.7680 \\
q_3(-2) &= \frac{1 + a(2) - 2b(2) - e^d}{e^u - e^d} = 0.6252
\end{aligned}$$

As $q_t(k) \in (0, 1)$, the market is complete. The binomial tree for the stock price in the martingale probability is as follows



The binomial tree for the interest rate is



The discount process in the martingale probability has the following distribution

$$D(0) = 1, \quad D(1) = \frac{1}{1 + R(0)} = 0.9709, \quad \text{with prob. } 1,$$

$$D(2) = \frac{D(1)}{1 + R(1)} = \begin{cases} \frac{D(1)}{1+0.07} = 0.9074, & \text{with prob. } q_1(0) \\ \frac{D(1)}{1+0.03} = 0.9426 & \text{with prob. } 1 - q_1(0) \end{cases},$$

$$D(3) = \frac{D(2)}{1 + R(2)} = \begin{cases} \frac{0.9074}{1+0.06} = 0.8560, & \text{with prob. } q_1(0)q_2(1) \\ \frac{0.9074}{1+0.04} = 0.8725 & \text{with prob. } q_1(1)(1 - q_2(1)) \\ \frac{0.9426}{1+0.04} = 0.9063 & \text{with prob. } (1 - q_1(0))q_2(-1) \\ \frac{0.9426}{1+0.02} = 0.9241 & \text{with prob. } (1 - q_1(0))(1 - q_2(-1)) \end{cases}$$

Now assume that we want to compute the initial price of a call option on the stock with strike $K = 10$ and maturity $T = 3$. This price is given by

$$\Pi(0) = \widetilde{\mathbb{E}}[D(3)(S(3) - 10)_+],$$

where the expectation is in the martingale probability $q_t(k)$.

To compute this expectation we need the joint distribution in the risk-neutral probability of the random variables $D(3), S(3)$. Using our results above we find that this joint distribution is given as in the following table:

$D(3) \downarrow$, $S(3) \rightarrow$	12.3368	10.7251	9.3239	8.1059
0.8560	$q_1(0)q_2(1)q_3(2)$	$q_1(0)q_2(1)(1-q_3(2))$	0	0
0.8725	0	$q_1(0)(1-q_2(1))q_3(0)$	$q_1(0)(1-q_2(1))(1-q_3(0))$	0
0.9063	0	$(1-q_1(0))q_2(-1)q_3(0)$	$(1-q_1(0))q_2(-1)(1-q_3(0))$	0
0.9241	0	0	$(1-q_1(0))(1-q_2(-1))q_3(-2)$	$(1-q_1(0))(1-q_2(-1))(1-q_3(-2))$

We conclude that

$$\begin{aligned} \Pi(0) = {} & 0.8560[(12.3368 - 10)q_1(0)q_2(1)q_3(2) + (10.7251 - 10)q_1(0)q_2(1)(1 - q_3(2))] \\ & + 0.8725(10.7251 - 10)q_1(0)(1 - q_2(1))q_3(0) + 0.9063(1 - q_1(0))q_2(-1)q_3(0) = 1.4373. \end{aligned}$$