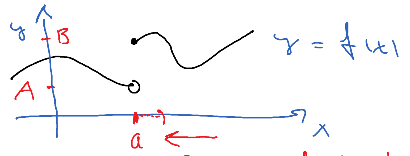
Gränsvärden: Om det minsta (ε) och det största - oändligheten (∞)

Fokus på räkneregler, egenskaper och typer:

$\frac{A}{B}$, $\frac{0}{0}$, $\frac{A}{0}$, $\frac{A}{\infty}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, $\frac{\infty}{B}$,

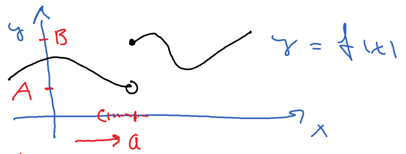
Gränsvärde: $\lim\limits_{x \to a} f(x)$, $\lim\limits_{x \to \infty} f(x)$, $\lim\limits_{x \to -\infty} f(x)$

Högergränsvärde: $\lim\limits_{x \to a^+} f(x) = B$



nära a från höger.

Vänstergränsvärde: $\lim\limits_{x \to a^-} f(x) = A$



nära a från vänster

Kontinuitet: $f(x)$ är kontinuerlig i $x = a$ $\in D_f$ om $\lim\limits_{x \to a} f(x) = f(a)$

Räkneregler: Om $\lim f(x) = A$ och $\lim g(x) = B$ så gäller

1) $\lim (f(x) + g(x)) = \lim f(x) + \lim g(x)$
2) $\lim (f(x) - g(x)) = \lim f(x) - \lim g(x)$
3) $\lim f(x) \cdot g(x) = \lim f(x) \cdot \lim g(x)$
4) $\lim f(x)/g(x) = \lim f(x) / \lim g(x)$ (om $B \neq 0$)

$$\lim \Leftrightarrow \begin{cases} \lim\limits_{x \to a}, \lim\limits_{x \to a^+}, \lim\limits_{x \to a^-} \\ \lim\limits_{x \to \infty} \\ \lim\limits_{x \to -\infty} \end{cases}$$

5) Sammansatta gränsvärden

$$\lim_{x \to a} f(g(x)) = \left[\begin{matrix} t = g(x) \to b \\ \text{då } x \to a \end{matrix} \right] = \lim_{t \to b} f(t)$$

$\lim\limits_{x \to a} g(x) = b$

$$\begin{cases} a \Leftrightarrow a^+, a^-, \infty, -\infty \\ b \Leftrightarrow b^+, b^-, \infty, -\infty \end{cases}$$

6) Olikheter:

a) $f(x) < g(x) \implies \lim\limits_{x \to a} f(x) \leq \lim\limits_{x \to a} g(x)$

b) $f(x) \leq h(x) \leq g(x) \implies \lim\limits_{x \to a} f(x) \leq \lim\limits_{x \to a} h(x) \leq \lim\limits_{x \to a} g(x)$

$A = \lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) \implies \lim\limits_{x \to a} h(x) = A$

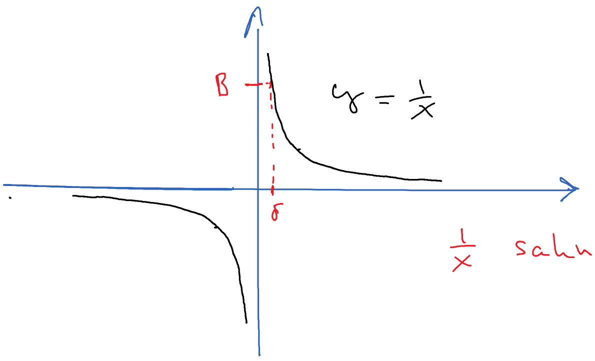
$\boxed{a \Leftrightarrow a^+, a^-, \infty, -\infty}$

Standardgränsvärden:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \;,\quad \lim_{x\to 0}\frac{1-\cos x}{x} = 0 \qquad \left(\text{Av typen } \frac{0}{0}\right)$$

$$\lim_{x\to 0^+}\frac{1}{x} = \infty \;,\quad \lim_{x\to 0^-}\frac{1}{x} = -\infty \qquad \left(\frac{1}{0+} \text{ respektive } \frac{1}{0^-}\right)$$

$$\lim_{x\to \infty}\frac{1}{x} = 0 \;,\quad \lim_{x\to -\infty}\frac{1}{x} = 0 \qquad \left(\frac{1}{\infty} \text{ respektive } \frac{1}{-\infty}\right)$$

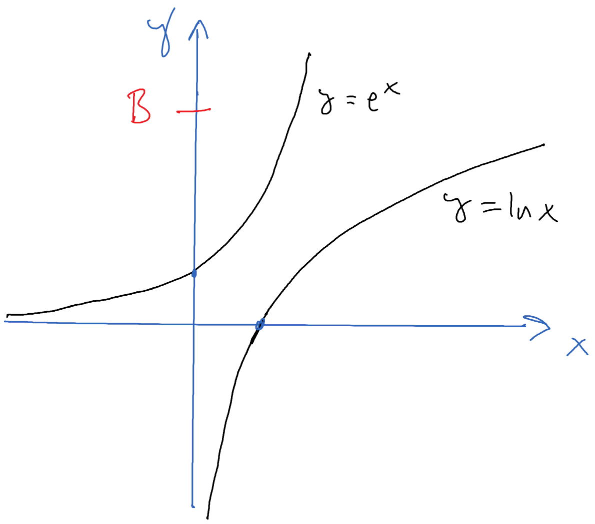


Givet $B > 0$ tag $\delta = \frac{1}{B}$

Då gäller

$0 < x < \delta \implies \frac{1}{x} > B$

$\frac{1}{x}$ saknar begränsning för x nära 0



$$\lim_{x\to 0^+}\ln x = -\infty \;,\quad \lim_{x\to \infty}\ln x = \infty$$

$$\lim_{x\to \infty} e^x = \infty \;,\quad \lim_{x\to -\infty} e^x = 0$$

Oändlighet (∞) har vi där en begränsning saknas

Uppgift 1

den 8 december 2020 00:04

Beräkna

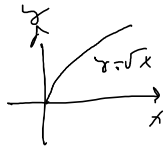
a) $\lim_{x\to\infty} \frac{4+x^2-3x^5}{4x^5+2x^3-x} = "\frac{-\infty}{\infty}"$

b) $\lim_{x\to\infty} \sqrt{x^2+3x} - \sqrt{x^2-1} = "\infty - \infty"$

$\lim_{x\to\infty} x^n = \infty \;,\; n>0$

$\lim_{x\to\infty} \frac{1}{x^n} = 0$ om $n>0$

$\lim_{x\to\infty} \sqrt{x} = \infty$



Lösning:

a) Termen med störst exponent domineras:

$$\lim_{x\to\infty} \frac{4+x^2-3x^5}{4x^5+2x^3-x} = \lim_{x\to\infty} \frac{\cancel{x^5}\left(\frac{4}{x^5}+\frac{1}{x^3}-3\right)}{\cancel{x^5}\left(4+\frac{2}{x^2}-\frac{1}{x^4}\right)} = \lim_{x\to\infty} \frac{\frac{4}{x^5}+\frac{1}{x^3}-3}{4+\frac{2}{x^2}-\frac{1}{x^4}} = \frac{0+0-3}{4+0-0}$$

$$= -\frac{3}{4}\,.$$

b) $\lim_{x\to\infty} \sqrt{x^2+3x} - \sqrt{x^2-1} =$ [Multiplicera med konjugerade uttrycket $\sqrt{x^2+3x} + \sqrt{x^2-1}$, konjugatregeln!]

$$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+3x} - \sqrt{x^2-1}\right)\left(\sqrt{x^2+3x} + \sqrt{x^2-1}\right)}{\sqrt{x^2+3x} + \sqrt{x^2-1}} = \left[(a-b)(a+b) = a^2-b^2\right]$$

$$= \lim_{x\to\infty} \frac{x^2+3x-(x^2-1)}{\sqrt{x^2+3x}+\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{3x+1}{\sqrt{x^2+3x}+\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{x\cdot\left(3+\frac{1}{x}\right)}{\sqrt{x^2\left(1+\frac{3}{x}\right)}+\sqrt{x^2\left(1-\frac{1}{x^2}\right)}}$$

$$= \lim_{x\to\infty} \frac{\cancel{x}\cdot\left(3+\frac{1}{x}\right)}{\cancel{x}\cdot\sqrt{1+\frac{3}{x}}+\cancel{x}\sqrt{1-\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{3+\frac{1}{x}}{\sqrt{1+\frac{3}{x}}+\sqrt{1-\frac{1}{x^2}}} = \frac{3+0}{\sqrt{1+0}+\sqrt{1-0}} = \frac{3}{1+1} = \frac{3}{2}$$

$\sqrt{x^2} = x$ men om $x<0$ så är $\sqrt{x^2} = -x$

Breakout 1

den 8 december 2020 00:09

Beräkna

$$\lim_{x\to 0} \cos x = \cos 0 = 1$$

a) $$\lim_{x\to\infty} \frac{2x+3}{\sqrt{4x^2-7x+1}} = \text{"}\frac{\infty}{\infty}\text{"}$$

b) $$\lim_{x\to-\infty} \frac{2x+3}{\sqrt{4x^2-7x+1}} = \text{"}\frac{-\infty}{\infty}\text{"}$$

$$\left(\sqrt{x^2} = |x|\right)$$

c) $$\lim_{x\to 0} \frac{\tan 4x}{x^2+x} = \text{"}\frac{0}{0}\text{"}$$

$$\left(\lim_{x\to 0} \frac{\sin x}{x} = 1\right)$$

Lösning: a) & b)

$$\frac{2x+3}{\sqrt{4x^2-7x+1}} = \frac{x\left(2+\frac{3}{x}\right)}{\sqrt{x^2\cdot\left(4-\frac{7}{x}+\frac{1}{x^2}\right)}} = \frac{x\cdot\left(2+\frac{3}{x}\right)}{\sqrt{x^2}\sqrt{4-\frac{7}{x}+\frac{1}{x^2}}} = \left[\sqrt{x^2} = |x| = \begin{cases} x, & x>0 \\ -x, & x<0\end{cases}\right.$$

a) $$= \frac{\cancel{x}\left(2+\frac{3}{x}\right)}{\cancel{x}\sqrt{4-\frac{7}{x}+\frac{1}{x^2}}} = \frac{2+\frac{3}{x}}{\sqrt{4-\frac{7}{x}+\frac{1}{x^2}}} \longrightarrow \frac{2+0}{\sqrt{4-0+0}} = \frac{2}{\sqrt{4}} = 1$$

då $x\to\infty$

b) $$= \frac{\cancel{x}\left(2+\frac{3}{x}\right)}{-\cancel{x}\sqrt{4-\frac{7}{x}+\frac{1}{x^2}}} \longrightarrow \frac{2+0}{-\sqrt{4-0+0}} = -1$$ då $x\to-\infty$

c) $$\lim_{x\to 0} \frac{\tan 4x}{x^2+x} = \lim_{x\to 0} \frac{\sin 4x}{(\cos 4x)\cdot x\cdot(x+1)} = \lim_{x\to 0} \frac{\sin 4x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos 4x}\cdot\lim_{x\to 0}\frac{1}{x+1}$$

$$= \lim_{x\to 0} \frac{\sin 4x}{x}\cdot\frac{1}{1}\cdot\frac{1}{0+1} = \lim_{x\to 0} 4\cdot\frac{\sin 4x}{4x} =$$

$$= 4\cdot\lim_{x\to 0}\frac{\sin 4x}{4x} = \left[\begin{matrix} t = 4x \to 0 \\ \text{då } x\to 0\end{matrix}\right] = 4\cdot\lim_{t\to 0}\frac{\sin t}{t} = 4\cdot 1 = \underline{\underline{4}}$$

Uppgift 2

den 8 december 2020 00:13

Beräkna

c) $\lim_{x\to 2} \dfrac{\sqrt{4x+1}-\sqrt{2x+5}}{4-2x} = "\dfrac{0}{0}"$

d) $\lim_{x\to 0} \dfrac{\tan x^2}{x\cdot \sin 7x} = "\dfrac{0}{0}"$

Lösning

c) " Lite tankar : $\left|\begin{array}{lll} \sqrt{4x+1} \to \sqrt{8+1} = 3 & \text{då} & x\to 2 \\ \sqrt{2x+5} \to \sqrt{4+5} = 3 & - " - & \\ 4-2x \to 4-4 = 0 & - " - & \end{array}\right\|$

$\lim_{x\to 2} \dfrac{\sqrt{4x+1}-\sqrt{2x+5}}{4-2x} = \left[\text{multipl. med konjugat: } '\sqrt{\ } + \sqrt{\ }'\right]$

$= \lim_{x\to 2} \dfrac{(\sqrt{4x+1}-\sqrt{2x+5})(\sqrt{4x+1}+\sqrt{2x+5})}{(4-2x)(\sqrt{4x+1}+\sqrt{2x+5})} = \lim_{x\to 2} \dfrac{4x+1-(2x+5)}{(4-2x)\cdot(\sqrt{\ }+\sqrt{\ })} =$

$= \lim_{x\to 2} \dfrac{2x-4}{(4-2x)(\sqrt{\ }+\sqrt{\ })} = \lim_{x\to 2} \dfrac{-1}{\sqrt{4x+1}+\sqrt{2x+5}} = \dfrac{-1}{3+3} = \underline{\underline{-\dfrac{1}{6}}}$

d)

$\lim_{x\to 0} \dfrac{\tan x^2}{x\cdot \sin 7x} = \lim_{x\to 0} \dfrac{\sin x^2}{(\cos x^2)\cdot x\cdot \sin 7x} = \left[\begin{array}{l} \cos x^2 \to \cos 0 = 1 \\ \text{då } x\to 0 \end{array}\right] =$

$= \underbrace{\lim_{x\to 0} \dfrac{1}{\cos x^2}}_{= 1/\cos 0 = 1} \cdot \lim_{x\to 0} \dfrac{\sin x^2}{x\cdot \sin 7x} = \lim_{x\to 0} \left(\dfrac{\sin x^2}{x^2} \cdot \dfrac{x}{\sin 7x}\right) =$

$\overset{!}{=} \lim_{x\to 0} \dfrac{\sin x^2}{x^2} \cdot \lim_{x\to 0} \dfrac{x}{\sin 7x} \overset{*}{=} 1\cdot \dfrac{1}{7} = \underline{\underline{\dfrac{1}{7}}}$

*)

1) $$\lim_{x\to 0}\frac{\sin x^2}{x^2} = \left[\begin{matrix} t = x^2 \to 0 \\ \text{då } x \to 0\end{matrix}\right] = \lim_{t\to 0}\frac{\sin t}{t} = \underline{\underline{1}}$$

2) $$\lim_{x\to 0}\frac{x}{\sin 7x} = \lim_{x\to 0}\frac{1}{\frac{7\cdot\sin 7x}{7x}} = \frac{1}{7}\cdot\frac{1}{\lim\limits_{x\to 0}\frac{\sin 7x}{7x}} = \left[\begin{matrix} t = 7x \to 0 \\ \text{då } x \to 0\end{matrix}\right] =$$

$$= \frac{1}{7}\cdot\frac{1}{\lim\limits_{t\to 0}\frac{\sin t}{t}} = \frac{1}{7}\cdot\frac{1}{1} = \underline{\underline{\frac{1}{7}}} \qquad \left(\lim_{x\to 0}\frac{x}{\sin x} = 1\right)$$

$z^2 = -15 + 8i$, Bestäm z.

Lösning Sätt $z = x + iy$

$z^2 = x^2 - y^2 + i\,2xy$

$= -15 + 8i$

Identifiera koeff.

$x^2 - y^2 = -15$

$2xy = 8$

Hjälpekv: $x^2 + y^2 = |-15 + 8i| = \sqrt{(-15)^2 + 8^2} = \sqrt{225 + 64} = \sqrt{289} = 17$

$2x^2 = -15 + 17 = 2$

$x^2 = 1, \quad x = \pm 1$

$y = \frac{8}{2x} = \frac{4}{x} = \pm 4$

Svar

$z_1 = 1 + 4i$

$z_2 = -1 - 4i$

$$\begin{array}{r} 17 \\ \cdot\ 17 \\ \hline 119 \\ 17 \\ \hline 289 \end{array}$$