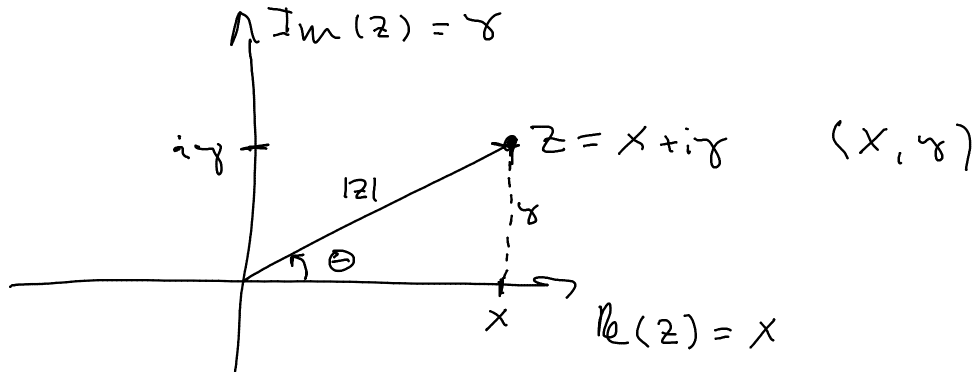
Agenda

den 24 november 2020 09:15

- <u>Komplexa tal</u>

$$z = x + iy \quad , \quad x, y \in \mathbb{R}$$

$$i^2 = -1$$



$$|z| = \sqrt{x^2 + y^2}$$

$$\arg(z) = \theta$$

De "vanliga" räknereglerna gäller:

$$z_1 = x_1 + iy_1 \quad , \quad z_2 = x_2 + iy_2$$

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 - z_2 = (x_1 + x_2) - i(y_1 + y_2)$$

$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) =$ ("korsvis multiplikation där $i^2 = -1$) $= \ldots$

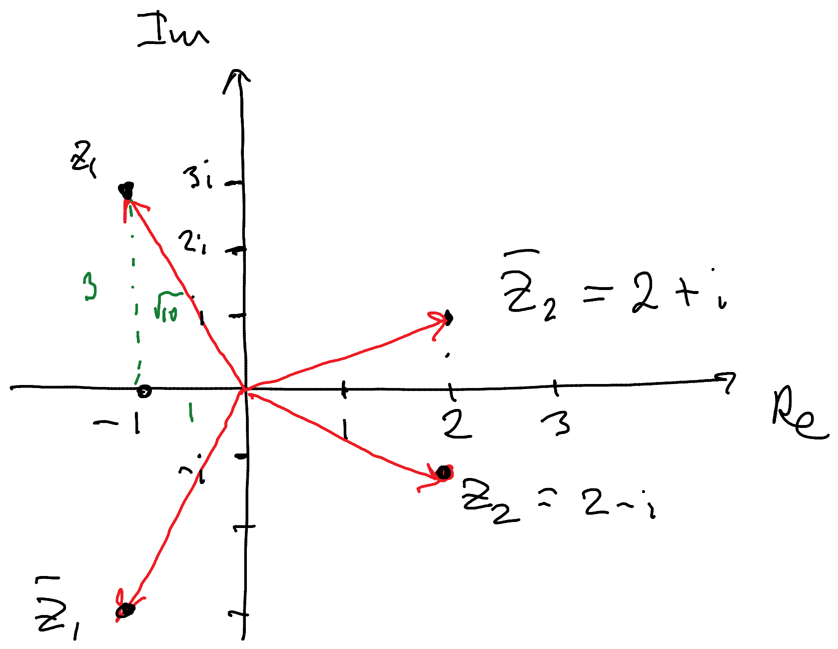
$$\frac{z_1}{z_2} = \frac{z_1 \cdot \bar{z}_2}{|z_2|^2} \qquad (\bar{z}_2 = x_2 - iy_2)$$

$$|z|^2 = z \cdot \bar{z}$$

Uppgift 1

den 24 november 2020 09:16

Antag $z_1 = -1 + 3i$, $z_2 = 2 - i$



$\bar{z}_1 = -1 - 3i$, $\bar{z}_2 = 2 + i$

Beräkna

a) $z_1 \cdot z_2$

b) $|z_1|$, $|z_2|$

c) $z_1 \cdot \bar{z}_1$, $z_2 \cdot \bar{z}_2$

d) $\dfrac{z_1}{z_2}$

Lösning:

a) $z_1 \cdot z_2 = (-1 + 3i) \cdot (2 - i) = -1 \cdot 2 + (-1) \cdot (-i) + 3i \cdot 2 + 3i \cdot (-i) =$

$= -2 + i + 6i - 3i^2 = (i^2 = -1) = -2 + 7i \underbrace{- 3 \cdot (-1)}_{+3} = \underline{\underline{1 + 7i}}$

b) $|z_1| = \sqrt{(-1)^2 + 3^2} = \sqrt{1 + 9} = \sqrt{10}$ $(= |\bar{z}_1|)$

$|z_2| = \sqrt{2^2 + 1^2} = \sqrt{4 + 1} = \sqrt{5}$ $(= |\bar{z}_2|)$

c) $z_1 \cdot \bar{z}_1 = (-1+3i)(-1-3i) = (-1)\cdot(-1) + (-1)(-3i) + 3i\cdot(-1) + 3i\cdot(-3i)$

$= 1 + \underbrace{3i - 3i}_{=0} - 9\underbrace{i^2}_{=-1} = 1+9 = 10 \quad (= |z_1|^2)$

$z_2 \cdot \bar{z}_2 = (2-i)(2+i) = \ldots = 4+1 = \underline{5} \quad (= |z_2|^2)$

d) $\dfrac{z_1}{z_2} = \dfrac{-1+3i}{2-i} = \dfrac{(-1+3i)\cdot(2+i)}{(2-i)\cdot(2+i)} = \dfrac{-1\cdot 2 + (-1)\cdot i + 3i\cdot 2 + 3i\cdot i}{\underline{\underline{5}}}$

$= \dfrac{-2-i+6i+3i^2}{5} = \dfrac{-5+5i}{5} = \underline{\underline{-1+i}}$

Breakout 1

den 24 november 2020 09:16

Antag $z_1 = -4+i$, $z_2 = -1+2i$

Beräkna

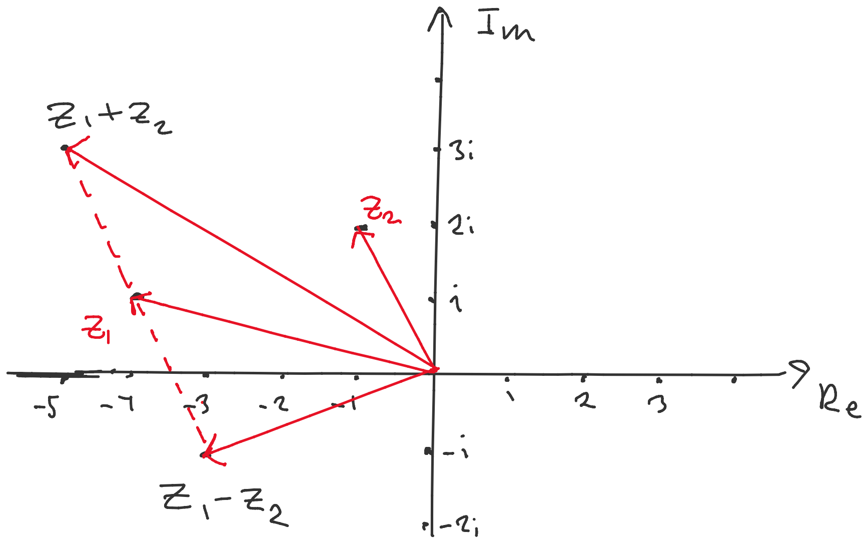
a) $z_1 + z_2$

b) $z_1 - z_2$

c) $\dfrac{z_1}{z_2}$

} Rita in dessa i det komplexa talplanet

d) Vad är $Re\left(\dfrac{z_1}{z_2}\right)$, $Im\left(\dfrac{z_1}{z_2}\right)$?



a) $z_1 + z_2 = -5+3i$

b) $z_1 - z_2 = -3-i$

c) $$\frac{z_1}{z_2} = \frac{-4+i}{-1+2i} = \frac{(-4+i)(-1-2i)}{(-1+2i)(-1-2i)} = \frac{4+8i-i-2i^2}{(-1)^2+(-2)^2} \overset{!}{=} \frac{6+7i}{1+4} = \frac{6+7i}{5} = \underset{}{\overset{\text{Re}\downarrow}{\frac{6}{5}}} + i\overset{\text{Im}\downarrow}{\frac{7}{5}}$$

d) $Re\left(\dfrac{z_1}{z_2}\right) = \dfrac{6}{5}$, $Im\left(\dfrac{z_1}{z_2}\right) = \dfrac{7}{5}$

Uppgift 2

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Lös ekvationerna

a) $(2-i)\bar{z} = 7-i$

b) $z^2+(2-2i)z+5+10i=0$

$z = x+iy$, $\bar{z} = x-iy$

$\bar{\bar{z}} = \overline{x-iy} = x+iy = z$.

Lösning

a) $(2-i)\bar{z} = 7-i$

$\bar{z} = \dfrac{7-i}{2-i} = \ldots = 3+i$

$\underline{\underline{z = 3-i}}$

b) $z^2+(2-2i)z+5+10i=0$ (*)

$\left(z+\frac{2-2i}{2}\right)^2 - \left(\frac{2-2i}{2}\right)^2 + 5+10i = 0$

$(z+1-i)^2 + 2i + 5 + 10i = 0$

(2) $(z+1-i)^2 + 5+12i = 0$

(3) $(z+1-i)^2 = -5-12i$

*

1) Dela med koefficienten framför z^2.

2) Kvadratkomplettera VL.

3) Skriv ekvationen på formen $w^2 =$ komplett tal

4) Sätt $w = x+iy$ och bestäm x och y

$w^2 = x^2-y^2+i2xy$.

$\left(\frac{2-2i}{2}\right)^2 = (1-i)^2 = 1^2 - 2\cdot 1\cdot i + (-i)^2$

$= 1-2i-1 = \underline{\underline{-2i}}$

Vi söker komplext tal $w = z+1-i$ sådant att $w^2 = -5-12i$

(4) Sätt $w = x+iy$ då är $w^2 = (x+iy)^2 = x^2 + i2xy + (iy)^2$
$= x^2 - y^2 + i \cdot 2xy$

Dvs, då är

$x^2 - y^2 + i \cdot 2xy = -5-12i$

(5) Dvs
1) $x^2 - y^2 = -5$
2) $2xy = -12$, $xy = -6$, $y = \frac{-6}{x}$

(6) Extra ekvation
3) $x^2 + y^2 = |-5-12i| = \sqrt{169} = 13$

Addera ekvationerna 1) & 3)

$2x^2 = -5 + 13 = 8$

(7) $x^2 = 4$, $x = \pm 2$

5. Identifiera real och imaginärdelar

6. Litet trick ger en extra ekvation:
$|w^2| = |-5-12i| = \sqrt{25+144}$
$= |w|^2$ $= \sqrt{169} = 13$
$= x^2 + y^2$

7. Lös ut x^2 & x

8. Bestäm y

9. Bestäm lösningarna z:
$z = w - (x+iy)$

Insättning i 2) ger

(8) Fall 1. $x = 2$, $xy = -6$, $2y = -6$, $y = -3$

Fall 2. $x = -2$, $xy = -6$, $-2y = -6$, $y = 3$

Vi får två lösningar

$$\begin{cases} w_1 = 2 - 3i \\ w_2 = -2 + 3i \end{cases}$$

(9) Dvs $z_1 = w_1 - (1-i) = 2 - 3i - 1 + i = \underline{1 - 2i}$

$z_2 = w_2 - (1-i) = -2 + 3i - 1 + i = \underline{-3 + 4i}$

Svar: $\begin{cases} z_1 = 1 - 2i \\ z_2 = -3 + 4i \end{cases}$

Breakout 2

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Lös ekvationen

$$iz^2 = 6 + 8i$$

Lösning Dela med i

$$z^2 = 8 - 6i$$

Sätt $z = x + iy$

$$z^2 = x^2 - y^2 + i \cdot 2xy = 8 - 6i$$

Identifiera realdelar & imaginärdelar

$$\begin{cases} 1) \; x^2 - y^2 = 8 \\ 2) \; 2xy = -6 \end{cases}$$

$$\frac{6}{i} = \frac{6i}{i \cdot i} = \frac{6i}{-1} = -6i$$

$$\left(\frac{1}{i} = -i\right)$$

Om $z = x + iy$ så är

$$z^2 = (x+iy)(x+iy) =$$

$$= x^2 + ixy + iyx + (iy)\cdot(iy)$$

$$= x^2 + i\,2xy + i^2 \cdot y^2 =$$

$$= x^2 - y^2 + i \cdot 2xy .$$

$$|z^2| = \sqrt{(x^2-y^2)^2 + (2xy)^2} =$$

$$= \sqrt{(x^2)^2 - 2x^2y^2 + (y^2)^2 + 4x^2y^2}$$

$$= \sqrt{(x^2)^2 + 2x^2y^2 + (y^2)^2}$$

$$= \quad ^2 + y^2)^2$$

$$= x^2 + y^2 = |z|^2$$

Extra ekvation

3) $x^2+y^2 = |z^2| = |8-6i| = \sqrt{8^2+(-6)^2} = \sqrt{\underbrace{64+36}_{=100}} = 10$

Vi adderar ekvationerna 1) & 3)

$$x^2-y^2+x^2+y^2 = 8+10$$

$$2x^2 = 18$$

$$x^2 = 9$$

$$x = \pm 3$$

Insättning i ekvation 2) ger

Fall 1: $x=3$; $2xy=-6$

$2\cdot 3\cdot y=-6$, $\underline{\underline{y=-1}}$

Fall 2: $x=-3$; $2xy=-6$

$2\cdot(-3)\cdot y=-6$, $\underline{\underline{y=1}}$

$\underline{\underline{\text{Svar}}}$

$$\begin{cases} z_1 = 3-i \\ z_2 = -3+i \end{cases}$$