Uppgift 1/Agenda
den 26 november 2020 12:28

Polär form / Polära koordinater

Skriv följande tal i polär form

a) $1+\sqrt{3}\,i$

b) $2-2i$

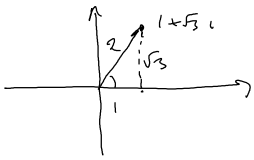
c) $-\sqrt{3}-i$

a) $z = 1+\sqrt{3}\,i$

$r = |z| = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{1+3} = 2$

$z = 2\cdot\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 2\cdot(\cos\theta + i\sin\theta)$

Dvs $\begin{cases}\cos\theta = \frac{1}{2}\\ \sin\theta = \frac{\sqrt{3}}{2}\end{cases}$



$\theta = \frac{\pi}{3}$

$z = 2\cdot\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

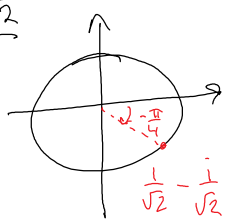
$= 2e^{i\frac{\pi}{3}}$

b) $w = 2-2i = |w|(\cos\varphi + i\sin\varphi) = |w|e^{i\varphi}$

$|w| = \sqrt{2^2+(-2)^2} = \sqrt{8} = 2\sqrt{2}$

$w = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - i\cdot\frac{1}{\sqrt{2}}\right)$

$= 2\sqrt{2}\,e^{-i\frac{\pi}{4}}$



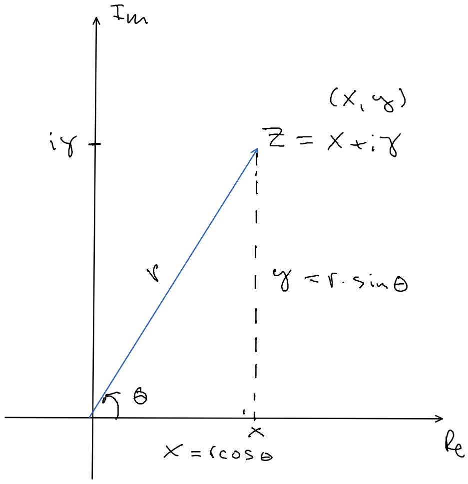
$= \cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}$

$\varphi = \arctan\frac{-1/\sqrt{2}}{1/\sqrt{2}} = \arctan(-1)$

$z\cdot w = 2\cdot\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\cdot$
$\cdot\, 2\sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$

$= 2\cdot 2\sqrt{2}\left(\cos\left(\frac{\pi}{3} - \frac{-\pi}{4}\right) + i\sin\left(\frac{\pi}{3} - \frac{-\pi}{4}\right)\right)$

$= 4\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right) = 4\sqrt{2}\,e^{i\frac{7\pi}{12}}$



$\arg(z) = \theta$

$z = r\cos\theta + i\cdot r\sin\theta =$
$= r(\cos\theta + i\sin\theta) \leftrightarrow (r,\theta)$ polära koord

$|z| = r = \sqrt{x^2+y^2}$

Sats Om $z = |z|(\cos\theta + i\sin\theta)$
$w = |w|(\sin\varphi + i\sin\varphi)$

så är

$z\cdot w = |z||w|(\cos(\theta+\varphi) + i\sin(\theta+\varphi))$

$z/w = (|z|/|w|)(\cos(\theta-\varphi) + i\sin(\theta-\varphi))$

Följdsats: de Moivres formel

$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$
$(n\in\mathbb{Z})$

Sats/Definition Eulers formel

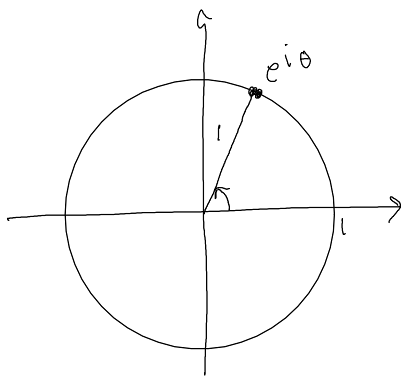
$e^{i\theta} = \cos\theta + i\sin\theta$

Egenskaper

$|e^{i\theta}| = 1$

$e^{i\theta}\cdot e^{i\varphi} = e^{i(\theta+\varphi)}$

$(e^{i\theta})^n = e^{in\theta}$

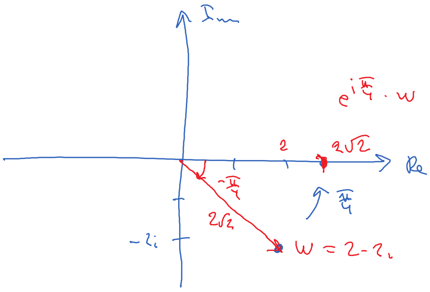


Ex

$$e^{i\frac{\pi}{4}} \cdot w = \left[\begin{array}{l}\text{vi roterar } w \\ \frac{\pi}{4} \text{ rad. motsols}\end{array}\right]$$

Argumentet ökar med $\frac{\pi}{4}$

$$= 1 \cdot 2\sqrt{2} \cdot e^{i\frac{\pi}{4}} \cdot e^{-i\frac{\pi}{4}} = 2\sqrt{2}\, e^{i \cdot 0} = 2\sqrt{2}$$



Breakout 1

den 1 december 2020 09:35

a) Skriv

$z = 3 - i\cdot 3\sqrt{3}$

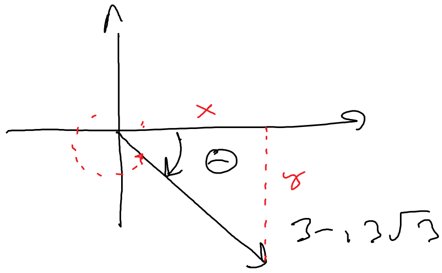
på polär form

b) Beräkna

$(3 - i3\sqrt{3})^{76}$

och skriv svaret på formen $x+iy$

Lösning:



a) $|z| = \sqrt{3^2 + (3\sqrt{3})^2} = \sqrt{9+9\cdot 3} = \sqrt{36} = 6$

$$z = 6\cdot\left(\frac{3}{6} - i\frac{3\sqrt{3}}{6}\right) = 6\cdot\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= 6\cdot(\cos\theta + i\sin\theta), \quad \theta = \arctan\frac{-3\sqrt{3}}{3} = \arctan(-\sqrt{3}) = -\frac{\pi}{3}$$

$$= 6\cdot e^{-i\pi/3} = 6\cdot\left(\cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3}\right)$$

b) $$z^{76} = \left(6\cdot e^{-i\pi/3}\right)^{76} = 6^{76}\cdot e^{-i76\cdot\frac{\pi}{3}}$$

$\left[76 = 3\cdot 25 + 1, \quad \frac{76}{3} = 25 + \frac{1}{3}\right]$

$$= 6^{76}\cdot e^{-i(25\pi + \frac{\pi}{3})} = 6^{76}\cdot e^{-i(\overbrace{24\pi}^{12\cdot 2\pi} + \pi + \frac{\pi}{3})} = 6^{76}\cdot e^{-i(\pi + \frac{\pi}{3})}$$

$$= 6^{76}\cdot \underbrace{e^{-i\pi}}_{=-1}\cdot e^{-i\frac{\pi}{3}} = -6^{75}\cdot 6e^{-i\frac{\pi}{3}} =$$

$\left[e^{-i\pi} = \underbrace{\cos(-\pi)}_{=-1} + i\underbrace{\sin(-\pi)}_{=0} = -1\right]$

$$= -6^{75}\cdot(3 - i3\sqrt{3})$$

$\left[6^{76} = 6^{75}\cdot 6^{1}\right]$

$\left[e^{-i\frac{\pi}{3}} = \frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$

$\left[e^{i\cdot 2\pi} = \underbrace{\cos 2\pi}_{=1} + i\underbrace{\sin 2\pi}_{=0} = 1\right]$

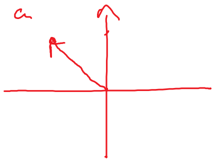
$\left[e^{-i24\pi} = 1\right]$

Breakout 2

den 1 december 2020 09:50

Lös ekvationen

$$z^4 = -1 + i$$



Beskriv lösningarna i en figur i komplexa talplanet.

Binomisk ekvation:

$z^n = w \qquad (n=4,\ w=-1+i)$

1. Skriv om w i polär form
   $w = \rho \cdot e^{i\varphi}$, $\varphi = \arg(w)$, $\rho = |w|$
2. Skriv z på polär form
   $z = r \cdot e^{i\theta}$, $r = |z|$
   $z^n = r^n \cdot (e^{i\theta})^n = r^n e^{in\theta}$ $\qquad e^{i(\varphi + k \cdot 2\pi)}$
3. $z^n = w \iff r^n e^{in\theta} = \rho e^{i\varphi}$
   $r = \sqrt[n]{\rho}$, $n\theta = \varphi + k \cdot 2\pi$, $k = 0, 1, 2, \ldots n-1$

Lösning

1) $w = -1 + i = \sqrt{2}\left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2} \cdot e^{i\varphi}$

$|w| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ $\qquad \begin{cases} \cos\varphi = \frac{-1}{\sqrt{2}} \\ \sin\varphi = 1/\sqrt{2} \end{cases}$

$\varphi \overset{\pi +}{=} \arctan \frac{1}{-1} \overset{\pi +}{=} \arctan(-1) = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$

$w = \sqrt{2}\, e^{i 3\pi/4}$

2) Sätt $z = re^{i\theta}$, $z^4 = r^4 e^{i4\theta}$ $\qquad r = |z|$, $r^4 = |z|^4 = |z^4|$

Dvs $\quad r^4 e^{i4\theta} = \sqrt{2}\, e^{i3\pi/4} \iff r^4 = \overbrace{\sqrt{2}}^{2^{1/2}}$ och $4\theta = \frac{3\pi}{4} + k \cdot 2\pi$

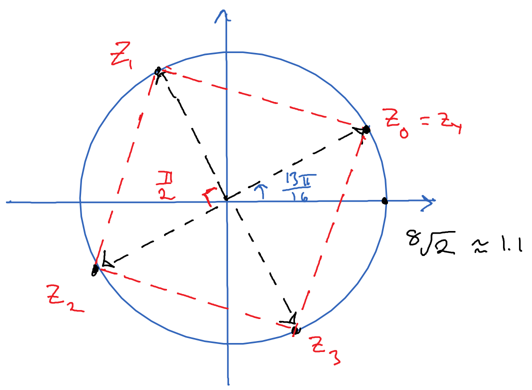
$r = \sqrt[4]{\sqrt{2}} = 2^{1/8} = \sqrt[8]{2}$ och $\theta_k = \frac{3\pi}{16} + k \cdot \frac{2\pi}{4} = \frac{3\pi}{16} + k \cdot \frac{\pi}{2}$, $k = 0, 1, 2, 3$

4 lösningar.

Dvs $\quad z_k = \sqrt[8]{2}\, e^{i\theta_k} = \sqrt[8]{2}\, e^{i\left(\frac{3\pi}{16} + k \cdot \frac{\pi}{2}\right)}$, $k = 0, 1, 2, 3$

ligger på en cirkel med radien $\sqrt[8]{2}$, $|z_k| = \sqrt[8]{2}$

i en regelbunden fyrhörning (n-hörning) $\qquad \frac{3\pi}{16} \approx 34^\circ$



Uppgift 2

den 1 december 2020 09:50

Lös andragradsekvationen

$$(1+i)z^2 + 2z - 3 + 11i = 0$$

Lösning: 1. Dela med $1+i$:

$$\frac{2}{1+i} = \frac{2\cdot(1-i)}{(1+i)(1-i)} = \frac{2-2i}{1^2+1^2} = 1-i$$

$$\frac{-3+11i}{1+i} = \frac{(-3+11i)(1-i)}{(1+i)(1-i)} = \frac{-3+3i+11i-11\underbrace{i^2}_{=-1}}{2} = \frac{8+14i}{2} = 4+7i$$

Dvs $z^2 + (1-i)z + 4 + 7i = 0$

$$\left[\left(\frac{1-i}{2}\right)^2 = \frac{1-2i+i^2}{4} = -\frac{i}{2}\right]$$

2. Kvadratkomplettera:

$$\left(z+\frac{1-i}{2}\right)^2 - \left(\frac{1-i}{2}\right)^2 + 4 + 7i = 0 \iff (\underbrace{z+\frac{1-i}{2}}_{=w})^2 + 4 + \frac{15i}{2} = 0$$

3. Sätt $w = z + \frac{1-i}{2} = x + iy$, $w^2 = (x+iy)^2 = x^2 - y^2 + i\cdot 2xy$

Dvs $w^2 = x^2 - y^2 + i\cdot 2xy = -4 - \frac{15i}{2}$

4. Identifiera Realdelar och Imaginärdelar: $\mathrm{Re}(VL) = \mathrm{Re}(HL)$, $\mathrm{Im}(VL) = \mathrm{Im}(HL)$

a) $x^2 - y^2 = -4$

b) $2xy = -\frac{15}{2}$

Hjälpekvation : $|w^2| = |-4-\frac{15}{2}i| = \sqrt{16+\frac{225}{4}} = \sqrt{\frac{289}{4}} = \frac{17}{2}$

$= |w|^2 = x^2 + y^2$

c) $x^2 + y^2 = \frac{17}{2}$

5. Lös ekv. systemet:

Addera ekvationerna a) & b) :

$$2x^2 = -4 + \frac{17}{2} = \frac{-8+17}{2} = \frac{9}{2}$$

$$x^2 = \frac{9}{4} \iff x = \pm\sqrt{\frac{9}{4}} = \pm\frac{3}{2}$$

Sätt in i b) och bestäm y:

$$2xy = -\frac{15}{2}$$

Två fall: $x = \frac{3}{2}$ ger $2 \cdot \frac{3}{2} \cdot y = \frac{-15}{2} \iff 3y = -\frac{15}{2}$, $y = -\frac{5}{2}$

$x = -\frac{3}{2}$. - - - $y = \frac{5}{2}$

<u>Dvs</u> $w_1 = \frac{3}{2} - i\frac{5}{2}$, $w_2 = -\frac{3}{2} + i\frac{5}{2}$

6.) Bestäm z:

$z = w - \frac{1-i}{2}$; $z_1 = \frac{3}{2} - i\frac{5}{2} - \frac{1-i}{2} = \underline{\underline{1-2i}}$

$z_2 = -\frac{3}{2} + i\frac{5}{2} - \frac{1-i}{2} = \underline{\underline{-2+3i}}$

Uppgift 2

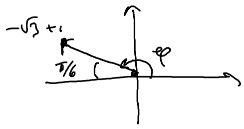
den 1 december 2020 09:49

Lös ekvationen $z^6 = -\sqrt{3} + i$

Lösning

1. Skriv $w = -\sqrt{3} + i$ på polär form: $|w| = \sqrt{3+1} = 2$, $w = 2 \cdot \left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

2:a



$$\begin{cases} \cos\varphi = -\frac{\sqrt{3}}{2} \\ \sin\varphi = \frac{1}{2} \end{cases}, \quad \varphi = \frac{5\pi}{6}$$

eller $\varphi = \pi + \arctan\frac{1}{-\sqrt{3}} = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

2. Skriv z på polär form $z = re^{i\theta}$

Vi får

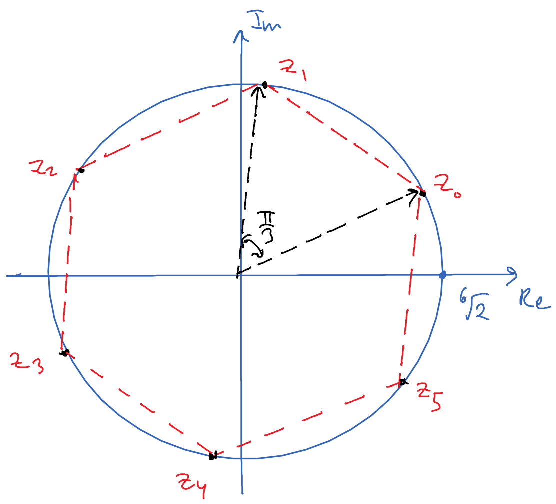
$$z^6 = r^6 e^{i6\theta} = 2 \cdot e^{i\frac{5\pi}{6}}$$

3. Identifiera r och θ

$r^6 = 2$ och $6\theta = \frac{5\pi}{6} + k\,2\pi$

$r = \sqrt[6]{2}$ och $\theta_k = \frac{5\pi}{36} + k\frac{2\pi}{6} = \frac{5\pi}{36} + k \cdot \frac{\pi}{3}$, $k = 0, 1, 2, 3, 4, 5$ (6 lösningar)

Dvs $z_k = \sqrt[6]{2}\, e^{i\left(\frac{5\pi}{36} + k \cdot \frac{\pi}{3}\right)}$, $k = 0, 1, 2, 3, 4, 5$



$\frac{\pi}{3} = 60^\circ$

$\frac{5\pi}{36} = 25^\circ$, $k = 0$

$\frac{17\pi}{36} = 85^\circ$, $k = 1$

$\frac{29\pi}{36} = 145^\circ$, $k = 2$

$\frac{41\pi}{36} = 205^\circ$, $k = 3$

$\frac{53\pi}{36} = 265^\circ$, $k = 4$

$\frac{65\pi}{36} = 325^\circ$, $k = 5$

Uppgift 3

den 1 december 2020 09:49

En triangel i planet har hörn i punkterna (0,0), (4,2) och (2,6)

Triangeln roteras 60° ($\frac{\pi}{3}$ rad) motsols kring (0,0)

Ange de nya koordinaterna för triangeln.

<u>Ledning:</u> Lägg triangeln i komplexa talplanet och multiplicera hörnen $(4,2) \leftrightarrow 4+2i$ och $(2,6) \leftrightarrow 2+6i$ med $e^{i\frac{\pi}{3}}$