Övningstenta

Anonym kod	MVE535/415 Matematisk Analys, Del 1 2019-03-21	Poäng

1. Till nedanstående uppgifter skall korta lösningar redovisas, samt svar anges, på anvisad plats (endast lösningar och svar på detta blad, och på anvisad plats, beaktas).

(a) Bestäm alla reella tal x sådana att $|5x+2| < 5$. (2 p)

Lösning: $|5x+2| < 5 \iff -5 < 5x+2 < 5 \iff$

$\iff -7 < 5x < 3 \iff -\frac{7}{5} < x < \frac{3}{5}$

Svar: $x \in \left(-\frac{7}{5}, \frac{3}{5}\right)$

(b) Lös ekvationen, (2 p)

$$2\log_3(x) + \log_9(x) = 10.$$

Lösning: Vet att: $\log_a(x) = \frac{\ln(x)}{\ln(a)} \Longrightarrow$

$\Rightarrow 2\log_3(x) + \log_9(x) = 2 \cdot \frac{\ln(x)}{\ln(3)} + \frac{\ln(x)}{\ln(9)} = 2\frac{\ln(x)}{\ln(3)} + \frac{\ln(x)}{\ln(3^2)} =$

$= 2 \cdot \frac{\ln(x)}{\ln(3)} + \frac{1}{2} \cdot \frac{\ln(x)}{\ln(3)} = \frac{5}{2} \cdot \frac{\ln(x)}{\ln(3)} = \frac{5}{2}\log_3(x) \overset{\text{vill}}{=} 10 \iff$

$\iff \log_3(x) = 4 \iff x = 3^4 = 81$

Svar: $x = 81$

(c) Beräkna gränsvärdet $\lim_{x\to-\infty} \frac{\sqrt{9x^2-5x+2}}{\sqrt[3]{8+x^3}}$ $(*)$ (2 p)

Lösning:

$$(*) = \lim_{x\to-\infty} \frac{\sqrt{x^2\left(9 - \frac{5}{x} + \frac{2}{x^2}\right)}}{\sqrt[3]{x^3\left(\frac{8}{x^3}+1\right)}} = \left\{ \begin{array}{l} \sqrt{x^2} = |x| \\ \sqrt[3]{x^3} = x \end{array} \right\} =$$

$$= \lim_{x\to-\infty} \frac{|x|\sqrt{9 - \frac{5}{x} + \frac{2}{x^2}}}{x\sqrt{\frac{8}{x^3}+1}} = \left\{ \begin{array}{l} x \to -\infty \\ \Rightarrow |x| = -x \end{array} \right\} = \lim_{x\to-\infty} -\frac{\sqrt{9 - \frac{5}{x} + \frac{2}{x^2}}}{\sqrt{\frac{8}{x^3}+1}} = -3$$

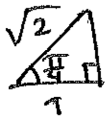
Svar: -3

(d) Bestäm $f'(\frac{1}{2})$ då $f(x) = \cos\big(\arctan(2x)\big)$. (2 p)

Lösning:

$$f'(x) = -\sin(\arctan(2x))\frac{1}{1+(2x)^2}\cdot 2 \Rightarrow$$

$$\Rightarrow f'(\tfrac{1}{2}) = -\sin(\arctan(1))\cdot\frac{1}{1+1^2}\cdot 2 = \Big\{$$



$$\Big\} = -\sin\left(\frac{\pi}{4}\right)\cdot\frac{1}{2}\cdot 2 = -\frac{1}{\sqrt{2}}$$

Svar: $-\frac{1}{\sqrt{2}}$

(e) Bestäm normallinjen till kurvan, (2 p)

$$2x + y - \sqrt{2}\sin(xy) = \frac{\pi}{2}$$

i punkten $(\frac{\pi}{4}, 1)$.

Lösning: Derivera implicit: $2+y'-\sqrt{2}\cos(xy)(y+xy') = 0$

Stoppa in $(\frac{\pi}{4},1)$: $2+y'-\sqrt{2}\cdot\frac{1}{\sqrt{2}}(1+\frac{\pi}{4}y') = 0 \Leftrightarrow$

$\Leftrightarrow 2+y'-1-\frac{\pi}{4}y' = 0 \Leftrightarrow (\frac{\pi}{4}-1)y' = 1$

$\Leftrightarrow y'\big|_{(\frac{\pi}{4},1)} = \frac{4}{\pi-4} \Rightarrow k_{Normal} = \frac{-1}{\frac{4}{\pi-4}} = \frac{4-\pi}{4}$

Går genom $(\frac{\pi}{4},1)$: $y = \frac{4-\pi}{4}(x-\frac{\pi}{4})+1$

Svar: $y = \frac{4-\pi}{4}x - \frac{\pi(4-\pi)}{16} + 1$

(f) Bestäm en primitiv funktion (antiderivata) till $f(x) = 4 + \sin(x) + \tan^2(x)$. (2 p)

Lösning:

$$f(x) = \underbrace{3}_{=(3x)'} + \underbrace{\sin(x)}_{=(-\cos(x))'} + \underbrace{1+\tan^2(x)}_{=(\tan(x))'}$$

$$\Rightarrow \int f(x)\,dx = 3x - \cos(x) + \tan(x) + C$$

Svar: $3x - \cos(x) + \tan(x)$

2. (a) $f'(x) = 7x^6 + 5x^4 + 3 > 0 \quad \forall x \in \mathbb{R} \Rightarrow$

$\Rightarrow f$ strängt växande på $\mathbb{R} \Rightarrow f$ injektiv på $\mathbb{R}$

$\Rightarrow f$ inverterbar.

Vet att : $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))} \Rightarrow$

$$\Rightarrow (f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \left\{ \begin{array}{l} f(1) = 5 \Rightarrow \\ \Rightarrow f^{-1}(5) = 1 \end{array} \right\} =$$

$$= \frac{1}{f'(1)} = \frac{1}{7+5+3} = \frac{1}{15}$$

(b) Låt $f^{-1}(x) = y \Rightarrow f(f^{-1}(x)) = f(y) \Leftrightarrow$

$$\Leftrightarrow x = \frac{y-a}{by-c} \Leftrightarrow x(by-c) = y-a \Leftrightarrow$$

$$\Leftrightarrow bxy - cx = y - a \Leftrightarrow y(bx-1) = cx - a$$

$$\Leftrightarrow y = f^{-1}(x) = \frac{cx-a}{bx-1}$$

$$f^{-1}(x) = f(x) \Leftrightarrow \frac{cx-a}{bx-1} = \frac{x-a}{bx-c}$$

Ser att om $c = 1$, så kan a, b vara vilka reella tal som helst, dvs $a, b \in \mathbb{R}$, $c = 1$

Finns det några fler lösningar?

$$\frac{cx-a}{bx-1} = \frac{x-a}{bx-c} \Leftrightarrow (cx-a)(bx-c) = (x-a)(bx-1)$$

$\Leftrightarrow \quad bcx^2 - c^2x - \cancel{ab}x + ac = bx^2 - x - \cancel{ab}x + a \quad \Leftrightarrow$

$\Leftrightarrow \quad \underbrace{b(c-1)}_{=0}x^2 + \underbrace{(1-c^2)}_{=0}x + \underbrace{a(c-1)}_{=0} = 0$

$$\Rightarrow \begin{cases} b(c-1) = 0 \\ a(c-1) = 0 \\ 1 - c^2 = 0 \end{cases}$$

Ser att vi även har lösningen

$a = 0, \quad b = 0, \quad c = -1$

3. f kontinuerlig $\forall x \in \mathbb{R} \setminus \{0, 1\}$

f kontinuerlig i $x = 0$ om:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\tan(4x)}{\sin(2x)} = \lim_{x \to 0^-} \frac{\frac{1}{\cos^2(4x)} \cdot 4}{\cos(2x) \cdot 2} = 2$$

$$\lim_{x \to 0^-} f(x) = f(0) \iff 2 = b \quad (*)$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x - 1}{(\sqrt{3x+1} - \sqrt{x+3})} \cdot \frac{\sqrt{3x+1} + \sqrt{x+3}}{(\sqrt{3x+1} + \sqrt{x+3})} =$$

$$= \lim_{x \to 1^+} \frac{(x-1)(\sqrt{3x+1} + \sqrt{x+3})}{3x + 1 - (x+3)} =$$

$$= \lim_{x \to 1^+} \frac{\cancel{(x-1)}(\sqrt{3x+1} + \sqrt{x+3})}{2\cancel{(x-1)}} = \frac{2+2}{2} = 2$$

$$\lim_{x \to 1^+} f(x) = f(1) \overset{(*)}{\iff} 2 = a + 2 \iff a = 0$$

$\therefore \quad a = 0, \quad b = 2$

4. <u>Steg 1</u>: $D_f = (0,\infty)\setminus\{1\} = (0,1)\cup(1,\infty)$

<u>Steg 2</u>: I. <u>Lodräta asymptoter</u>:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x}{\ln(x)} = -\infty$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x}{\ln(x)} = \infty$$

$\Longrightarrow$ $x = 1$ lodrät asymptot

II. <u>Vågräta asymptoter</u>:

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} \frac{x}{\ln(x)} \overset{\left[\frac{\infty}{\infty}\right]}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \infty$$

$$\left(\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x}{\ln(x)} = 0\right)$$

$\Longrightarrow$ Inga vågräta asymptoter

III. <u>Sneda asymptoter</u>:

$$k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1}{\ln(x)} = 0$$

$$m = \lim_{x\to\infty} \left(f(x) - 0\cdot x\right) = \lim_{x\to\infty} \frac{x}{\ln(x)} = \infty$$

$\Rightarrow$ Inga sneda asymptoter

<u>Steg 2</u>: $f'(x) = \dfrac{1\cdot \ln(x) - x\cdot\frac{1}{x}}{(\ln(x))^2} = \dfrac{\ln(x)-1}{(\ln(x))^2}$

$f'(x) = 0 \Rightarrow \ln(x) - 1 = 0 \Leftrightarrow \ln(x) = 1 \Leftrightarrow x = e$

Kritisk punkt

Steg 4: $$f''(x) = \frac{\frac{1}{x}\cdot(\ln(x))^2 - (\ln(x)-1)\cdot 2\ln(x)\cdot\frac{1}{x}}{(\ln(x))^4} =$$

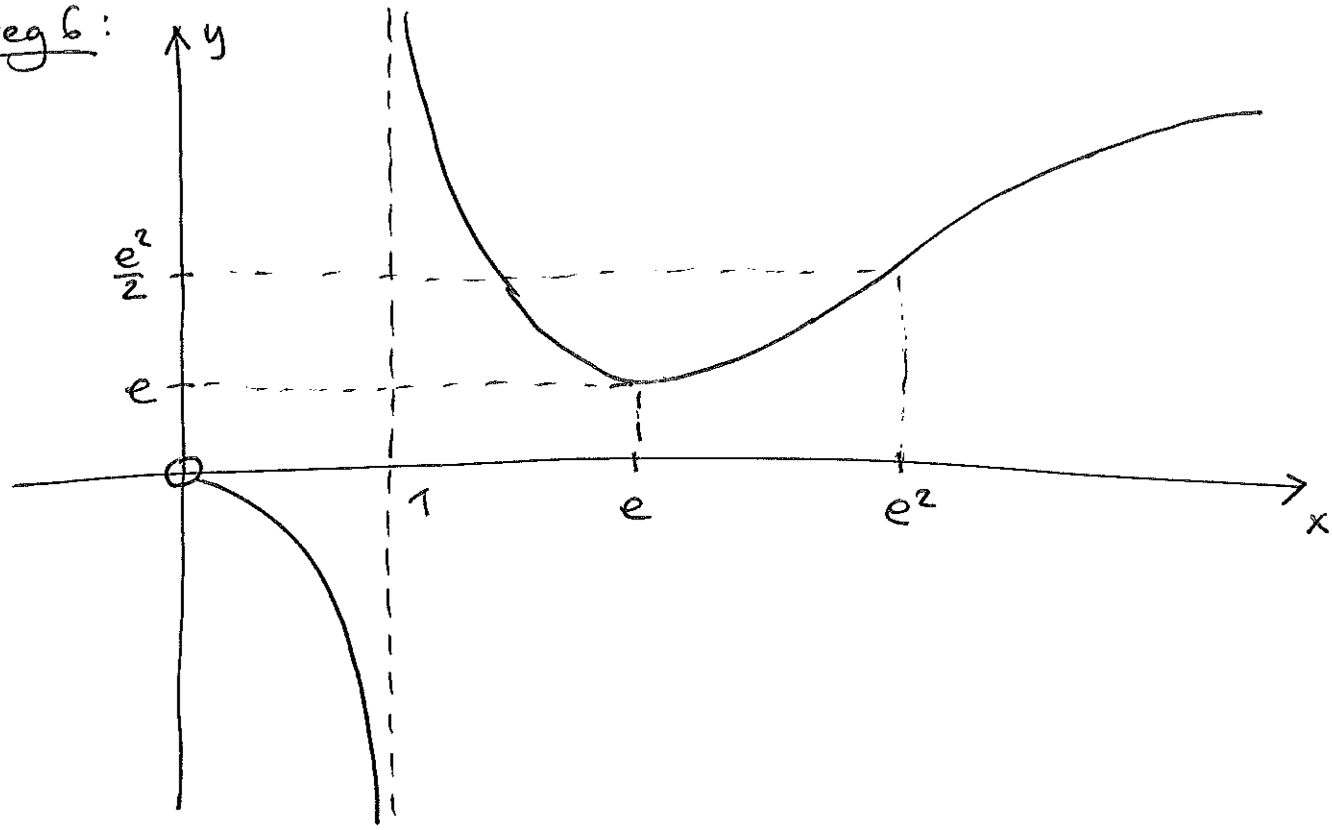
$$= \frac{\frac{1}{x}\cancel{\ln(x)}\left(\ln(x) - 2(\ln(x)-1)\right)}{(\ln(x))^{\cancel{4}3}} = \frac{2-\ln(x)}{x(\ln(x))^3}$$

$f''(x) = 0 \Rightarrow 2 - \ln(x) = 0 \Leftrightarrow x = e^2$ ev. infl. pkter.

Steg 5:

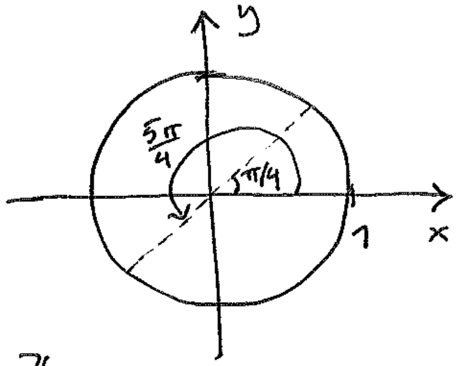
	0^+		1^-	1^+		e		e^2		∞
f'		$-$			$-$	0	$+$		$+$	
f''		$-$			$+$		$+$		$-$	
f	0	$\searrow\cap$	$-\infty$	∞	$\searrow\cup$	e	$\nearrow\cup$	$\frac{e^2}{2}$	$\nearrow\cap$	∞
						min. pkt.		infl. pkt.		

Steg 6:



5. $f'(x) = -e^{-x}\sin(x) + e^{-x}\cos(x) = e^{-x}(\cos(x) - \sin(x))$

Inga singulära punkter



$f'(x) = 0 \Rightarrow \cos(x) - \sin(x) = 0 \Leftrightarrow$

$\Leftrightarrow \tan(x) = 1 \Rightarrow$

$\Rightarrow x = \arctan(1) + n\pi = \frac{\pi}{4} + n\pi \,.\; n \in \mathbb{Z}$

$\Rightarrow x_1 = \frac{\pi}{4} \;,\; x_2 = \frac{5\pi}{4}$ Kritiska punkter

$f''(x) = -e^{-x}(\cos(x) - \sin(x)) + e^{-x}(-\sin(x) - \cos(x)) =$

$= -2e^{-x}\cos(x)$

$f''(x) = 0 \Rightarrow \cos(x) = 0 \Rightarrow x_3 = \frac{\pi}{2} \;,\; x_4 = \frac{3\pi}{2}$ ev. infl. pkter.

	0		$\frac{\pi}{4}$		$\frac{\pi}{2}$		$\frac{5\pi}{4}$		$\frac{3\pi}{2}$		2π
f'		+	0	−		−	0	+		+	
f''		−		−	0	+		+	0	−	
f		↗ ∩		↘ ∩		↘ ∪		↗ ∪		↘ ∩	

Kritiska punkter:

$f(\frac{\pi}{4}) = e^{-\frac{\pi}{4}}\sin(\frac{\pi}{4}) = \frac{e^{-\frac{\pi}{4}}}{\sqrt{2}} > 0$

$f(\frac{5\pi}{4}) = e^{-\frac{5\pi}{4}}\sin(\frac{5\pi}{4}) = e^{-\frac{5\pi}{4}}(-\sin(\frac{\pi}{4})) = -\frac{e^{-\frac{5\pi}{4}}}{\sqrt{2}} < 0$

Randpunkter:

$f(0) = 0 \;,\; f(2\pi) = 0$

$\therefore$ Globalt max. värde: $\frac{e^{-\frac{\pi}{4}}}{\sqrt{2}}$

Globalt min. värde: $-e^{-\frac{5\pi}{4}}/\sqrt{2}$

$x = \frac{\pi}{2}$ och $x = \frac{3\pi}{2}$ inflektionspunkter

6. $f(x) = \frac{1}{x} + 2\ln(x+1)$ ← väldef. om $x > -1$

↖ väldef. om $x \neq 0$

$\Rightarrow D_f = (-1, 0) \cup (0, \infty)$

Ser att:

$f(x) \to -\infty$ då $x \to -1^+$ eller $x \to 0^-$ (*)

$f(x) \to \infty$ då $x \to 0^+$ eller $x \to \infty$ (**)

(Obs! Detta innebär <u>inte</u> att $V_f = \mathbb{R}$!)

$$f'(x) = -\frac{1}{x^2} + \frac{2}{x+1}$$

$$f'(x) = 0 \Leftrightarrow \frac{2}{x+1} = \frac{1}{x^2} \Leftrightarrow 2x^2 = x + 1 \Leftrightarrow$$

$$\Leftrightarrow x^2 - \frac{1}{2}x - \frac{1}{2} = 0 \Rightarrow$$

$$\Rightarrow x = \frac{1}{4} \pm \sqrt{\frac{1}{16} + \frac{1}{2}} = \frac{1}{4} \pm \frac{3}{4} \Rightarrow x_1 = -\frac{1}{2}, \; x_2 = 1$$

(*) $\Rightarrow x_1 = -\frac{1}{2}$ lokalt max.

(**) $\Rightarrow x_2 = 1$ lokalt min.

$$f(-\tfrac{1}{2}) = -2 + 2\ln(\tfrac{1}{2}) = -2 - 2\ln(2) < 0$$

$$f(1) = 1 + 2\ln(2) > 0$$

$$\Rightarrow f(-\tfrac{1}{2}) < f(1)$$

$$\therefore V_f = (-\infty, -2 - 2\ln(2)] \cup [1 + 2\ln(2), \infty)$$

7. $f(x) = \arctan((x-1)^2)$. $D_f = \mathbb{R}$

Undersök tecknen för f' och f''

$$f'(x) = \frac{1}{1+((x-1)^2)^2} \cdot 2(x-1) = \frac{2(x-1)}{1+(x-1)^4}$$

$$f'(x) = 0 \Rightarrow 2(x-1) = 0 \Leftrightarrow x = 1$$

$$f''(x) = \frac{2\cdot(1+(x-1)^4) - 2(x-1)\cdot 4(x-1)^3}{(1+(x-1)^4)^2} =$$

$$= \frac{2 + 2(x-1)^4 - 8(x-1)^4}{(1+(x-1)^4)^2} = \frac{2-6(x-1)^4}{(1+(x-1)^4)^2}$$

$$f''(x) = 0 \Rightarrow 2 - 6(x-1)^4 = 0 \Leftrightarrow (x-1)^4 = \frac{2}{6} = \frac{1}{3}$$

$$\Rightarrow x - 1 = \pm \frac{1}{3^{1/4}} \Rightarrow$$

$$\Rightarrow x_1 = 1 - \frac{1}{3^{1/4}}, \quad x_2 = 1 + \frac{1}{3^{1/4}}$$

		$1-1/3^{1/4}$		1		$1+1/3^{1/4}$	
f'	$-$	$-$	$-$	0	$+$	$+$	$+$
f''	$-$	0	$+$	$+$	$+$	0	$-$

$\therefore$ f växande på $[1, \infty)$, avtagande på $(-\infty, 1]$

f konvex på $\left[1 - \frac{1}{3^{1/4}}, 1 + \frac{1}{3^{1/4}}\right]$

f konkav på $\left(-\infty, 1 - \frac{1}{3^{1/4}}\right] \cup \left[1 + \frac{1}{3^{1/4}}, \infty\right)$

8. Låt $f(x) = x^{1/x}$ och $g(x) = C$

Antal lösningar till ekvationen $x^{1/x} = C$

$\Longleftrightarrow$

Antal skärningspunkter mellan f och g:s grafer för olika värden på C.

Rita grafen till f!

<u>Steg 1</u>: $f(x) = x^{1/x} = e^{\ln(x^{1/x})} = e^{\frac{\ln(x)}{x}}$. $D_f = (0, \infty)$

<u>Steg 2</u>: $\lim\limits_{x \to 0^+} \frac{\ln(x)}{x} \overset{[\infty \cdot (-\infty)]}{=} -\infty \Longrightarrow$

$$\Rightarrow \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{\frac{\ln(x)}{x}} = 0 \quad \left(\text{"}e^{-\infty} = \frac{1}{\infty} = 0\text{"}\right)$$

$$\lim_{x \to \infty} \frac{\ln(x)}{x} \overset{[\frac{\infty}{\infty}]}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0 \Longrightarrow$$

$$\Rightarrow \lim_{x \to \infty} f(x) = \lim_{x \to \infty} e^{\frac{\ln(x)}{x}} = e^0 = 1$$

<u>Steg 3</u>: $f'(x) = e^{\frac{\ln(x)}{x}} \left(\frac{\frac{1}{x} \cdot x - 1 \cdot \ln(x)}{x^2} \right) = e^{\frac{\ln(x)}{x}} \cdot \frac{1 - \ln(x)}{x^2}$

$$f'(x) = 0 \Rightarrow 1 - \ln(x) = 0 \Longleftrightarrow x = e$$

<u>Steg 4</u>: f'' behövs ej!

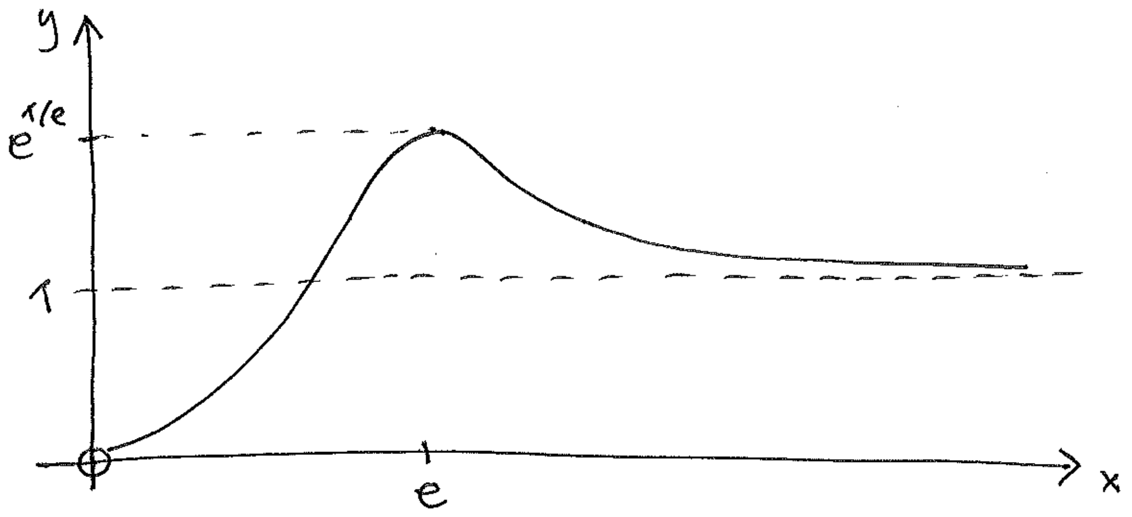
<u>Steg 5</u>:

	0		e		∞
f'		+		−	
f	0	↗	$\uparrow$ $e^{1/e}$	↘	1

$\Rightarrow x = e$ max-pkt. och vi borde ha $e^{1/e} > 1$

Kontroll: $e > 2 \Rightarrow e^{1/e} > 2^{1/e} > 1$ då $\frac{1}{e} > 0$ ok!

Steg 6:



Av figuren framgår att vi har:

1 lösning då $C \in (0, 1] \cup \{e^{1/e}\}$

2 lösningar då $C \in (1, e^{1/e})$

Inga lösningar då $C \in \mathbb{R} \setminus (0, e^{1/e}]$