Anonym kod	MVE535/415 Matematisk Analys, Del 1 2019-03-21	Poäng

1. Till nedanstående uppgifter skall korta lösningar redovisas, samt svar anges, på anvisad plats (endast lösningar och svar på detta blad, och på anvisad plats, beaktas).

(a) Bestäm alla reella tal x sådana att $\frac{x^2+1}{x-1} \leq x$. $(*)$ (2 p)

Lösning:

$$(*) \Leftrightarrow \frac{x^2+1}{x-1} - x \cdot \frac{x-1}{x-1} \leq 0 \Leftrightarrow \frac{x+1}{x-1} \leq 0$$

		-1		1	
$x+1$	$-$	0	$+$		$+$
$x-1$	$-$		$-$	0	$+$
$\frac{x+1}{x-1}$	$+$	0	$-$	ej def.	$+$

Svar: $x \in [-1, 1)$

(b) Förenkla (så långt som möjligt), (2 p)

$$2\log_{10}(5) - \frac{\ln(5)}{\ln(10)} - 2\log_{100}(50) + \ln(e^2) - e^{\ln(3)}. \quad (*)$$

Lösning:

$$(*) = \log_{10}(5^2) - \log_{10}(5) - 2\frac{\ln(50)}{\ln(100)} + 2 - 3 =$$

$\ln(100) = \ln(10^2) = 2\ln(10)$

$$= \log_{10}\left(\frac{5^2}{5}\right) - \frac{2}{2}\log_{10}(50) - 1 = \log_{10}\left(\frac{5}{50}\right) - 1 =$$

$$= \log_{10}(10^{-1}) - 1 = -1 - 1 = -2$$

Svar: -2

(c) Bestäm transversal- och brännpunkterna till ellipsen $2x^2 + 8y^2 = 72$. (2 p)

Lösning:

$$2x^2 + 8y^2 = 72 \Leftrightarrow \frac{2x^2}{72} + \frac{8y^2}{72} = 1 \Leftrightarrow \frac{x^2}{36} + \frac{y^2}{9} = 1 \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{x}{6}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \Rightarrow \text{transversalpunkter}: (\pm 6, 0)$$

$$c^2 = 6^2 - 3^2 = 36 - 9 = 27 \Rightarrow c = \pm\sqrt{27}$$

Svar: Transversalpunkter $= (\pm 6, 0)$, Brännpunkter $= (\pm\sqrt{27}, 0)$

(d) Beräkna derivatan av $f(x) = 4\sqrt{x} + 3$ med hjälp av derivatans definition. (2 p)

Lösning:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{4\sqrt{x+h} + 3 - (4\sqrt{x} + 3)}{h} =$$

$$= \lim_{h\to 0} \frac{4(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{4(x+h-x)}{h(\sqrt{x+h} + \sqrt{x})} =$$

$$= \frac{4}{\sqrt{x} + \sqrt{x}} = \frac{2}{\sqrt{x}}$$

Svar: $\frac{2}{\sqrt{x}}$

(e) Bestäm normallinjen till kurvan, (2 p)

$$y\cos(x) = 1 + \sin(xy)$$

i punkten $(0, 1)$.

Lösning: Derivera implicit:

$$y'\cos(x) - y\sin(x) = \cos(xy)(y + xy')$$

Stoppa in $(0,1)$: $y'|_{(0,1)} = 1 \implies k_{Normal} = -1$

Går genom $(0,1)$: $y = -1 \cdot (x - 0) + 1$

Svar: $y = -x + 1$

(f) Bestäm en primitiv funktion (antiderivata) till $f(x) = \dfrac{x^2}{\sqrt{x^3 + 4}}$. (2 p)

Lösning: Har att: $\frac{d}{dx}\sqrt{x^3+4} = \frac{3x^2}{2\sqrt{x^3+4}} = \frac{3}{2} \cdot \frac{x^2}{\sqrt{x^3+4}}$

$$\implies \int f(x)\,dx = \int \frac{x^2}{\sqrt{x^3+4}}\,dx = \frac{2}{3}\int \frac{3x^2}{2\sqrt{x^3+4}}\,dx =$$

$$= \frac{2}{3}\int \frac{d}{dx}\sqrt{x^3+4}\,dx = \frac{2}{3}\sqrt{x^3+4} + C$$

Svar: $\frac{2}{3}\sqrt{x^3+4}$

2 (a) $D\left(\frac{\ln(x^2-1)}{x}\right) = \frac{D(\ln(x^2-1))\cdot x - \ln(x^2-1)\cdot D(x)}{x^2} =$

$= \frac{\frac{1}{x^2-1}\cdot 2x^2 - \ln(x^2-1)}{x^2} = \frac{2}{x^2-1} - \frac{1}{x^2}\ln(x^2-1)$

(b) $D\left((\arcsin(x^2))^{-1}\right) = -(\arcsin(x^2))^{-2}\cdot D(\arcsin(x^2)) =$

$= -\frac{1}{(\arcsin(x^2))^2}\cdot\frac{1}{\sqrt{1-x^4}}\cdot D(x^2) =$

$= -\frac{2x}{\sqrt{1-x^4}\,(\arcsin(x^2))^2}$

(c) $D\left(\frac{2}{3}\arctan\left(\frac{1}{3}\tan\left(\frac{x}{2}\right)\right)\right) =$

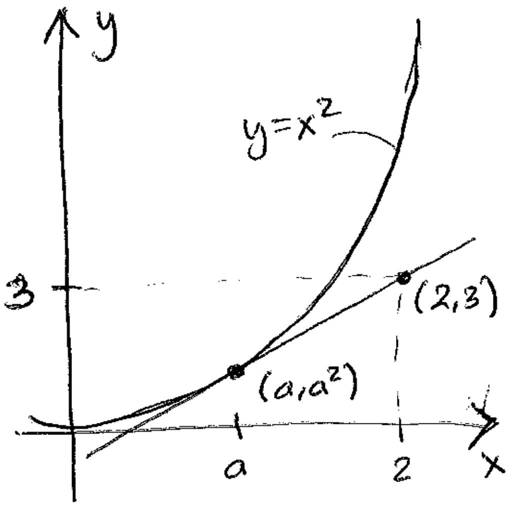
$= \frac{2}{3}\cdot\frac{1}{1+\left(\frac{1}{3}\tan\left(\frac{x}{2}\right)\right)^2}\cdot D\left(\frac{1}{3}\tan\left(\frac{x}{2}\right)\right) =$

$= \frac{2}{3}\cdot\frac{1}{1+\frac{1}{9}\tan^2\left(\frac{x}{2}\right)}\cdot\frac{1}{3}\cdot\frac{1}{\cos^2\left(\frac{x}{2}\right)}\cdot D\left(\frac{x}{2}\right) =$

$= \frac{\cancel{2}}{1}\cdot\frac{1}{\cos^2\left(\frac{x}{2}\right)}\cdot\frac{1}{9+\frac{\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)}}\cdot\frac{1}{\cancel{2}} = \frac{1}{9\cos^2\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)} =$

$= \frac{1}{8\cos^2\left(\frac{x}{2}\right)+1}$

3. Antag att punkten på kurvan $y=x^2$ har x-koordinaten a. Vi söker $y=kx+m$ s.a. $k=2x \; (=(x^2)')$ i denna punkt, dvs $k=2a$.



Därtill vill vi att tangenten ska gå genom punkterna (a,a^2) och $(2,3)$. Detta ger ekv. systemet:

$$\begin{cases} 3 = 2a\cdot 2 + m \\ a^2 = 2a\cdot a + m \end{cases} \iff \begin{cases} 3-4a = m \\ a^2 - 2a^2 = m \end{cases}$$

$$\Rightarrow 3-4a = -a^2 \iff a^2 - 4a + 3 = 0$$

$$\Rightarrow a = 2 \pm \sqrt{4-3} = 2 \pm 1 \Rightarrow a_1 = 1, \; a_2 = 3$$

$\underline{a=1}$: $k = 2a = 2$, $m = -a^2 = -1$

$\therefore \; y = 2x - 1$

$\underline{a=3}$: $k = 2a = 6$, $m = -a^2 = -9$

$\therefore \; y = 6x - 9$

4. Steg 1: $D_f = \mathbb{R}$

Steg 2: I. Lodräta: f definierad överallt

$\Rightarrow$ Inga lodräta asymptoter!

II. Vågräta:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3}{x^2\left(1+\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{x}{1+\frac{1}{x^2}} = \infty$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^3}{x^2\left(1+\frac{1}{x^2}\right)} = \lim_{x\to-\infty} \frac{x}{1+\frac{1}{x^2}} = -\infty$$

$\Rightarrow$ Inga vågräta asymptoter!

III. Sneda:

$$k_{1,2} = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{1}{1+\frac{1}{x^2}} = 1$$

$$m_{1,2} = \lim_{x\to\pm\infty} (f(x)-x) = \lim_{x\to\pm\infty} \left(\frac{x^3}{1+x^2} - x\cdot\frac{1+x^2}{1+x^2}\right) =$$

$$= \lim_{x\to\pm\infty} -\frac{x}{1+x^2} = \lim_{x\to\pm\infty} -\frac{1}{x\left(1+\frac{1}{x^2}\right)} = 0$$

$\Rightarrow$ $y=x$ sned asymptot!

Steg 3: $$f'(x) = \frac{3x^2(x^2+1) - x^3\cdot 2x}{(x^2+1)^2} = \frac{x^2(x^2+3)}{(x^2+1)^2}$$

$f'(x)=0 \Rightarrow x=0$ kritisk punkt

Ser att $f'(x) \geq 0 \ \forall x \in D_f \Rightarrow f$ alltid växande

Steg 4: $$f''(x) = \frac{(4x^3+6x)(x^2+1)^2 - (x^4+3x^2)2(x^2+1)\cdot 2x}{(x^2+1)^4} =$$

$$= \frac{2x\cancel{(x^2+1)}\left((2x^2+3)(x^2+1) - 2(x^4+3x^2)\right)}{(x^2+1)^{\cancel{4}\,3}} =$$

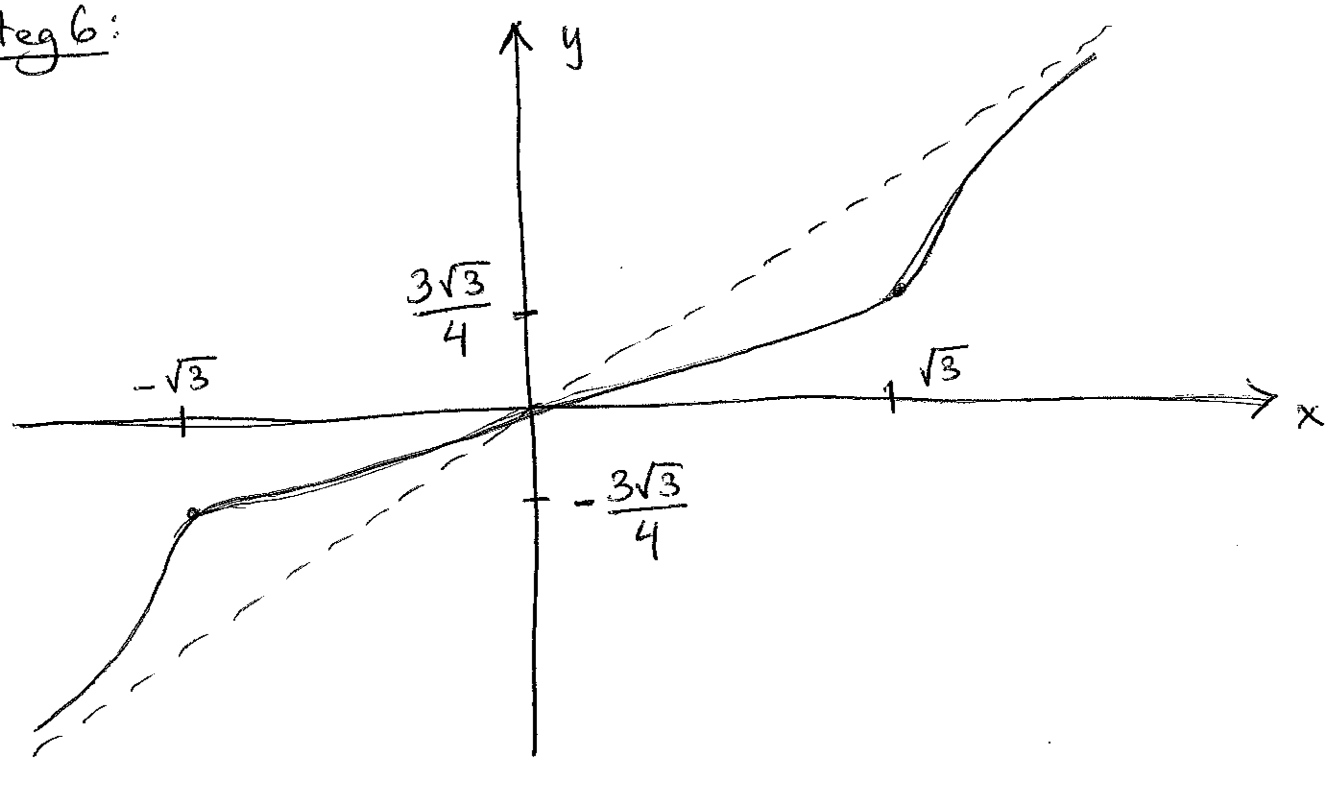
$$= \frac{2x(\cancel{2x^4} + 5x^2 + 3 - \cancel{2x^4} - 6x^2)}{(x^2+1)^3} = \frac{2x(3-x^2)}{(x^2+1)^3}$$

$f''(x) = 0 \implies x_1 = -\sqrt{3}\ ,\ x_2 = 0\ ,\ x_3 = \sqrt{3}$ ev. infl. pkter.

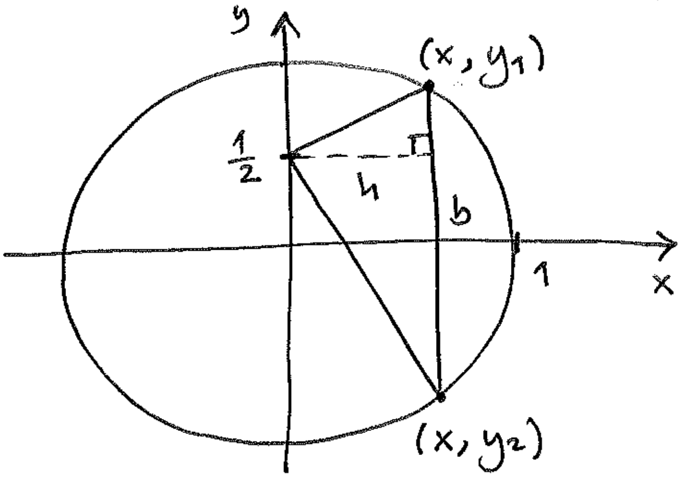
Steg 5:

	$-\infty$		$-\sqrt{3}$		0		$\sqrt{3}$		∞
f'		$+$		$+$	0	$+$		$+$	
f''		$+$	0	$-$	0	$+$	0	$-$	
f	$-\infty$	$\nearrow \cup$	$-\frac{3\sqrt{3}}{4}$	$\nearrow \cap$	0	$\nearrow \cup$	$\frac{3\sqrt{3}}{4}$	$\nearrow \cap$	∞
			infl.pkt.		infl.pkt.		infl.pkt.		

Steg 6:



5. Vi har följande figur



$$\text{Area} = \frac{b \cdot h}{2}$$

Vi ser att:

$h = x$

$b = y_1 - y_2$

Då (x, y_1) och (x, y_2) ligger på enhetscirkelns har vi att:

$$b = y_1 - y_2 = \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) = 2\sqrt{1-x^2}$$

$$\Rightarrow \text{Area} = A(x) = \frac{x \cdot 2\sqrt{1-x^2}}{2} = x\sqrt{1-x^2}, \quad D_A = [0, 1]$$

$$A'(x) = \sqrt{1-x^2} + x \cdot \frac{1}{2} \frac{-2x}{\sqrt{1-x^2}} =$$

$$= \frac{\sqrt{1-x^2} \cdot \sqrt{1-x^2}}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}$$

$$A'(x) = 0 \Rightarrow 1 - 2x^2 = 0 \Leftrightarrow x^2 = \frac{1}{2} \Rightarrow x = \overset{+}{(-)} \frac{1}{\sqrt{2}}$$

$$A(0) = A(1) = 0 \Rightarrow x = \frac{1}{\sqrt{2}} \text{ max. pkt.}$$

$$A\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \cdot \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \underline{\underline{\frac{1}{2}}} \text{ a.e.}$$

6. I en punkt x har tangenten till $y=\frac{6}{x^2+3}$ lutningen

$$y' = -\frac{12x}{(x^2+3)^2}$$

Låt $f(x) = -\frac{12x}{(x^2+3)^2}$. Vill minimera $f(x)$, $D_f = \mathbb{R}$.

$$f(x) = -\frac{12x}{\left(x^2\left(1+\frac{3}{x^2}\right)\right)^2} = -\frac{12x}{x^4\left(1+\frac{3}{x^2}\right)^2} \to 0 \text{ då } x \to \pm\infty$$

$$f'(x) = -\frac{12(x^2+3)^2 - 12x\cdot 2(x^2+3)\cdot 2x}{(x^2+3)^4} =$$

$$= -\frac{12\cancel{(x^2+3)}\left((x^2+3)-4x^2\right)}{(x^2+3)^{\cancel{4}3}} = -\frac{12(3-3x^2)}{(x^2+3)^3} =$$

$$= -\frac{12\cdot 3(1-x^2)}{(x^2+3)^3} = -\frac{36(1-x)(1+x)}{(x^2+3)^3}$$

$f'(x)=0 \Rightarrow x_1=-1,\ x_2=1$ kritiska punkter

	$-\infty$		-1		1		∞
f'		$+$	0	$-$	0	$+$	
f	0	$\nearrow$		$\searrow$		$\nearrow$	0

$f(-1) = \frac{12}{16} = \frac{3}{4} > 0 \Rightarrow x=-1$ global max.punkt

$f(1) = -\frac{12}{16} = -\frac{3}{4} < 0 \Rightarrow x=1$ global min.punkt

$y(1) = \frac{6}{1^2+3} = \frac{6}{4} = \frac{3}{2}$

$\therefore$ Tangenten till $y=\frac{6}{x^2+3}$ har minst lutning i punkten $\left(1, \frac{3}{2}\right)$.

7. Låt $f(x) = x^x$ och $g(x) = C$

Antal lösningar till ekvationen $x^x = C$

$\Longleftrightarrow$

Antal skärningspunkter mellan f och g:s grafer för olika värden på C.

Rita grafen till f!

Steg 1: $f(x) = x^x = e^{\ln(x^x)} = e^{x\ln(x)}$, $D_f = (0, \infty)$

Steg 2: $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x^x = \infty$

$$\lim_{x\to 0^+} \underbrace{x\cdot\ln(x)}_{[0\cdot(-\infty)]} = \lim_{x\to 0^+} \underbrace{\frac{\ln(x)}{\frac{1}{x}}}_{\left[\frac{-\infty}{\infty}\right]} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} =$$

$$= \lim_{x\to 0^+} -\frac{x^2}{x} = 0 \Rightarrow$$

$$\Rightarrow \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{x\cdot\ln(x)} = e^0 = 1$$

Steg 3: $f'(x) = e^{x\ln(x)}\left(1\cdot\ln(x) + x\cdot\frac{1}{x}\right) =$

$$= e^{x\ln(x)}(\ln(x) + 1)$$

$$f'(x) = 0 \Rightarrow \ln(x) + 1 = 0 \Leftrightarrow x = e^{-1}$$

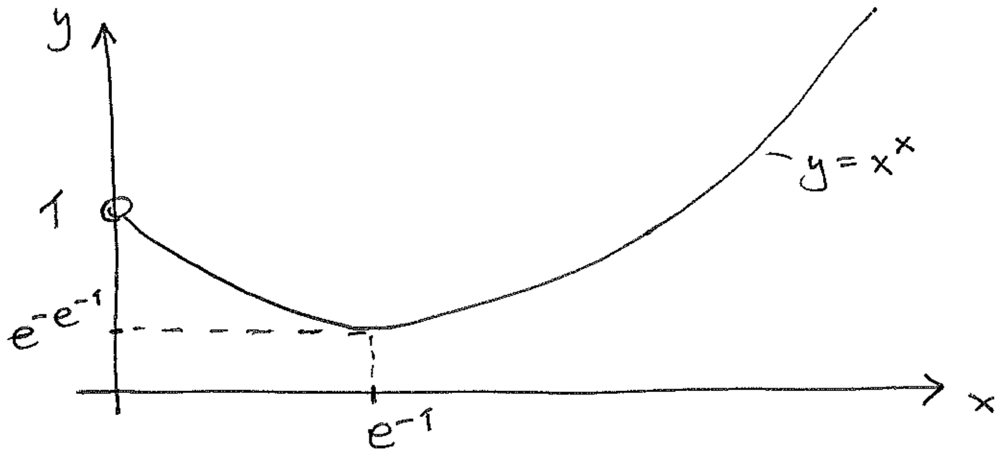
Steg 4: f'' behövs ej!

Steg 5:

	0^+		e^{-1}		∞
f'		$-$	0	$+$	
f	1	$\searrow$	min.	$\nearrow$	∞

$$f(e^{-1}) = e^{e^{-1}\cdot\ln(e^{-1})} = e^{-e^{-1}} < e^0 = 1$$

Steg 6:



Av figuren framgår att vi har:

1 lösning då $C \in \{e^{-e^{-1}}\} \cup [1, \infty)$

2 lösningar då $C \in (e^{-e^{-1}}, 1)$

Inga lösningar då $C \in (-\infty, e^{-e^{-1}})$

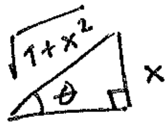
8 (a) <u>Definition</u>: $f: \mathbb{R} \to \mathbb{R}$ är <u>injektiv</u> om:

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2) \quad \forall x_1, x_2 \in \mathbb{R}$$

(b) <u>Bevis</u>: $\cos\left(\arctan\left(\sin\left(\frac{\pi}{2} - \arctan(x)\right)\right)\right) =$

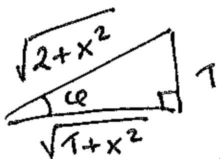
$$= \cos\left(\arctan\left(\overbrace{\sin\left(\tfrac{\pi}{2}\right)}^{=1}\cos(\arctan(x)) - \overbrace{\cos\left(\tfrac{\pi}{2}\right)}^{=0}\sin(\arctan(x))\right)\right) =$$

$$= \cos(\arctan(\cos(\arctan(x)))) = \Bigg\{$$



$\theta = \arctan(x)$ $\Bigg\} =$

$$= \cos\left(\arctan\left(\frac{1}{\sqrt{1+x^2}}\right)\right) = \Bigg\{$$



$\varphi = \arctan\left(\frac{1}{\sqrt{1+x^2}}\right)$ $\Bigg\} =$

$$= \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}} \quad \blacksquare$$