F4_2=8_alice

den 11 februari 2021 07:44

Mål

§ 3.7 Exempel av affina avbildningar

Kapitel 4: $V = \mathbb{R}^n$, span(C), Bas

$T: \mathbb{R}^n \to \mathbb{R}^m$ Matriser $A_{n \times m}$

(*) $A \cdot B = (Ab_1 \; Ab_2 \dots Ab_n)$

$B = (b_1 \; b_2 \dots b_n)$

Ex. 3.37

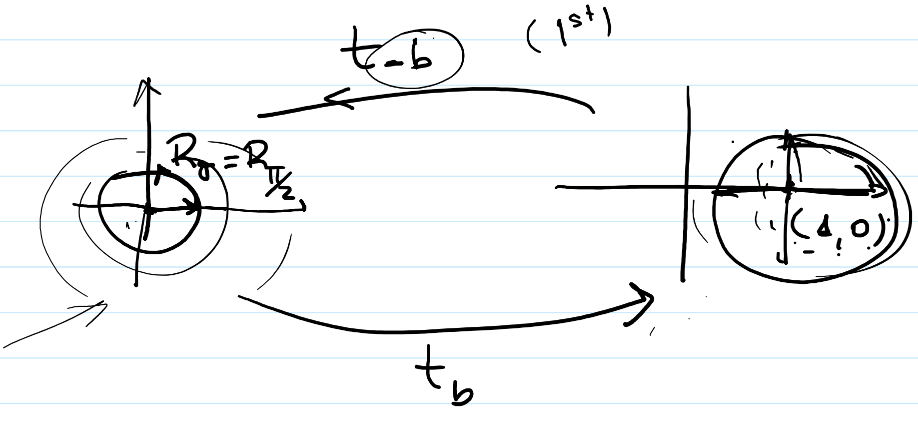
$f: T \to T$

$x \to f(x) = Ax + \cancel{b}$

linjär $(b = 0)$

$\underbrace{Ax + \underbrace{b}_{b \neq 0}}_{\text{Affin}}$

Hitta A, b som defineras den affin avbildning f som Roterar punkterna (vektor) i planet $\pi/2$ radianer moturs kring punkten $C = (1,0)$



$C \neq (0,0)$

$t_{-b}: \mathbb{R}^2 \to \mathbb{R}^2$

$u \to t_{-b}(u) = u - b = u - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{t_{-b}(u)}$

$R_{\pi/2}: \mathbb{R}^2 \to \mathbb{R}^2$

$u \to R_{\pi/2}(u) = \underbrace{\begin{pmatrix} \cos \pi/2 & -\sin \pi/2 \\ \sin \pi/2 & \cos \pi/2 \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix}$

$$u \longrightarrow R_{\pi/2}(u) = \underbrace{\begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$f(u) = t_b\left(R_{\frac{\pi}{2}}\left(\underbrace{t_{-b}(u)}\right)\right) = \left(t_b \circ R_{\pi/2} \circ t_{-b}\right)(u)$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = t_b\left(\underbrace{R_{\pi/2}\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)}\right) =$$

$$t_b\left(\underbrace{A\begin{bmatrix} x \\ y \end{bmatrix} - A\begin{bmatrix} 1 \\ 0 \end{bmatrix}}\right) =$$

$$\underbrace{A\begin{bmatrix} x \\ y \end{bmatrix}} \underbrace{- A\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{b}$$

$$A = \begin{pmatrix} \cos\pi/2 & -\sin\pi/2 \\ \sin\pi/2 & \cos\pi/2 \end{pmatrix} \qquad - A\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix}} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -y \\ x \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \underbrace{\begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

Exempel 3.38 sidan 108

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ affin avbildning

$u \longrightarrow f(u) = Au + b$ (? $A = ?$ $b = ?$)

sådant att f består av:

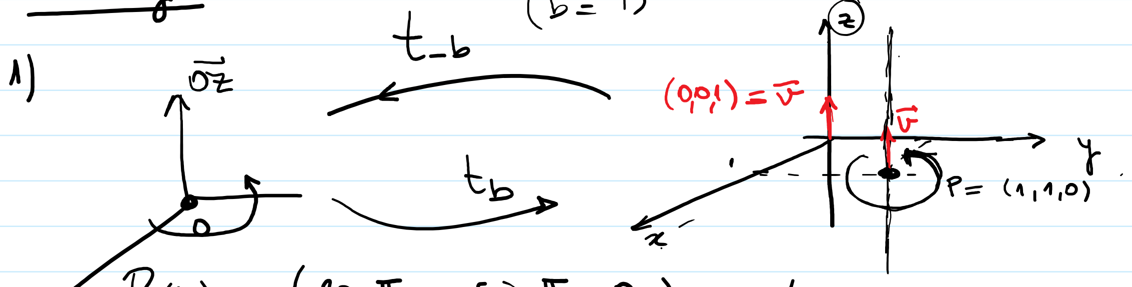
1) en rotation $\mathcal{R}$ ett halv varv ($\alpha = \pi$) runt linje L
2) följt av speglingen S i planet Π.

2) följt av speglingen S i planet Π.

$$L: \begin{pmatrix}x\\y\\z\end{pmatrix} = \underbrace{\begin{pmatrix}1\\1\\0\end{pmatrix}}_{P} + t\begin{pmatrix}0\\0\\1\end{pmatrix} \qquad \Pi: z = 1.$$

Lösning

1)



$$R(u) = \begin{pmatrix}\cos\pi & -\sin\pi & 0\\ \sin\pi & \cos\pi & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

R_{xy}

$$t_{-b}(u) = u - b = \underbrace{\begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix}}$$

$$R(t_{-b}(u)) = \underbrace{\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}}_{A}\left[\begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix}\right]$$

$$r_{\text{runt } L} = (t_b \circ R \circ t_{-b})(u) =$$

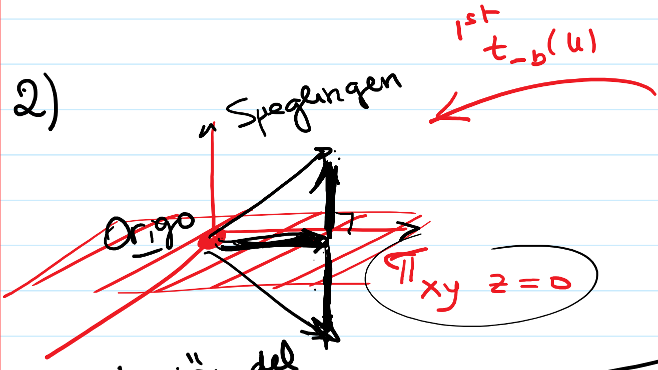
$$= t_b\left(A\left(\begin{bmatrix}x\\y\\z\end{bmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix}\right)\right) =$$

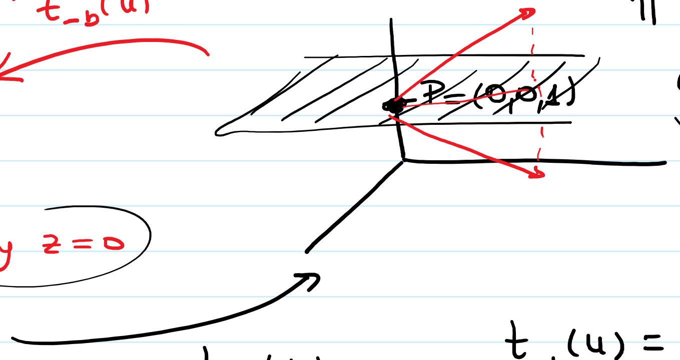
$$A\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$A\begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\2\\0\end{pmatrix}$$

$$r(u) = \underbrace{\left(A\begin{bmatrix}x\\y\\z\end{bmatrix} \underbrace{- A\begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}1\\1\\0\end{pmatrix}}_{\begin{pmatrix}2\\2\\0\end{pmatrix}}\right)}$$

2)





Π: $\boxed{z = 1}$

$$\underbrace{0x + 0y + 1z - 1 = 0}$$

$t_b(u)$

3:e del

$$t_{-b}(u) = u - \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$t_{-b}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$S(u) = (t_b \circ S \circ t_{-b})(u)$$

$$= t_b \circ S\Big(\overbrace{\begin{pmatrix}x\\y\\z\end{pmatrix}}^{u} - \overbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}^{b}\Big)$$

$$= t_b\left(\underbrace{\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}}_{S(u)} - \underbrace{\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}}_{S(b)}\right)$$

$$\boxed{S\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}}$$

$$= \left(S\begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}0\\0\\-1\end{pmatrix}\right) + \underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{b}$$

$$= S\begin{pmatrix}x\\y\\z\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix} =$$

$$= S\begin{pmatrix}x\\y\\z\end{pmatrix} + \begin{pmatrix}0\\0\\2\end{pmatrix}$$

3del

$$f(u) = S(r(u)) = (S \circ r)(u)$$

$$f = S\Big(\underbrace{R(u) + \begin{pmatrix}2\\2\\0\end{pmatrix}}_{\begin{pmatrix}x\\y\\z\end{pmatrix}}\Big)$$

$$f(u) = \underbrace{\overset{\text{matris}}{S}\,\overset{\text{matris}}{R}}_{A}u + S\begin{pmatrix}2\\2\\0\end{pmatrix} + \begin{pmatrix}0\\0\\2\end{pmatrix}$$

$$f(u) = A u + \begin{pmatrix}2\\2\\0\end{pmatrix} + \begin{pmatrix}0\\0\\2\end{pmatrix}$$

$$= A u + \begin{pmatrix}2\\2\\2\end{pmatrix}$$

$$= \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} + \begin{pmatrix}2\\2\\2\end{pmatrix}$$

$$= \underbrace{\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}}_{A}\begin{pmatrix}x\\y\\z\end{pmatrix} + \underbrace{\begin{pmatrix}2\\2\\2\end{pmatrix}}_{b}$$

$$f(u) = A \cdot u \quad b$$

Kapitel 4 Rummet $\mathbb{R}^n$

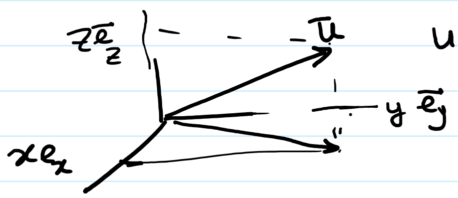
$V =$ vektorrummet $\mathbb{R}^n$

$V_1 = \mathbb{R}$ dimension $n=1$ $\quad x \in \mathbb{R}$, $x \cdot 1$

$V_2 = \mathbb{R}^2$ $\dim V\, n = 2$ $\quad u \in \mathbb{R}^2$ $\quad u = (x,y) = x(1,0) + y(0,1)$

$V_3 = \mathbb{R}^3$ $\dim V = n = 3$ $\quad u \in \mathbb{R}$

$u = (x,y,z) = x\vec{e_x} + y\vec{e_y} + z\vec{e_z}$



$V = \mathbb{R}^4, \mathbb{R}^5, \ldots, \mathbb{R}^n$

$(x_1 x_2 x_3 x_4) \quad \ldots \quad (x_1, x_2, \ldots x_n)$

$$V: \mathbb{R}^n = \left\{ v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \; v_1 \in \mathbb{R}, \ldots v_n \in \mathbb{R} \right\}$$

$$\left\{ v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}; \; v_i \in \mathbb{R}, \; i = 1, \ldots, n \right\}$$

$$\left\{ w = (w_1, w_2, \ldots, w_n), \; w_i \in \mathbb{R} \right\}$$

$V, +, \alpha \cdot \quad , \alpha \in \mathbb{R}$

$$u + v = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \left(u_i + v_i \right) \quad i = 1, \ldots, n$$

$$\alpha v = \alpha \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = (\alpha v_i) \quad i = 1, \ldots n$$

$$u \cdot v = \begin{pmatrix} u_1 \\ \vdots \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \end{pmatrix} = u_1 \cdot v_1 + u_2 \cdot v_2 + \ldots + u_n v_n =$$

$$u \cdot v = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \underbrace{u_1 \cdot v_1 + u_2 \cdot v_2 + \dots + u_n \cdot v_n}_{\sum_{k=1}^{n} u_k \cdot v_k} =$$

$$\|u\| = \sqrt{u \cdot u} = \sqrt{\sum_{k=1}^{n} u_k^2} = \sqrt{u_1^2 + \dots + u_n^2}$$

Def 4.3

- $u \parallel v$ u parallell med v

om det finns $\alpha \in \mathbb{R}$ $(\alpha \neq 0)$

$$\boxed{u = \alpha v}$$

- $u \perp v$ u ortogonal mot v

om och endast om

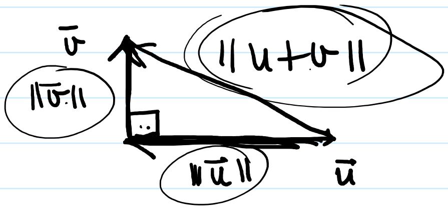
$u \cdot v = 0$

$$\boxed{\underbrace{\cos\alpha}_{-1 < \cos\alpha < 1} = \frac{u \cdot v}{\|u\| \cdot \|v\|}}$$

$$\frac{u \cdot v}{\|u\|\|v\|} \leq 1 \Rightarrow u \cdot v \leq \|u\| \cdot \|v\|$$

Cauchy-Schwartz olikhet.

Sats: 46 Pythagoras satsen



$$\underbrace{\|u+v\|^2} = (u+v) \cdot (u+v) = \underbrace{u \cdot u} + \underbrace{u \cdot v + v \cdot u} + v \cdot v$$

$$\|u+v\|^2 = \|u\|^2 + 2\underbrace{u \cdot v} + \|v\|^2$$

Om $u \cdot v = 0$ $(u \perp v)$

$$\boxed{\|u+v\|^2 = \|u\|^2 + \|v\|^2}$$

om $u \cdot v \neq 0$

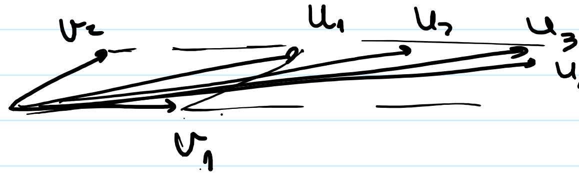
$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2\|u\|\|v\|\cos\alpha$ $\quad \alpha$ = vinkeln med u och v

Def 4.7 <u>linjärkombination</u>

u är linjärkombination av $v_1, v_2 \dots v_n$ om det finns $\alpha_1, \dots \alpha_n \in \mathbb{R}$ sådant att

$$\underline{u = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n}$$

$n=2$



$u_4 = \alpha v_1 + \beta v_2$

$$\mathrm{Span}(V) = \{ w = \alpha_1 v_1 + \dots + \alpha_n v_n \,/\, \alpha_i \in \mathbb{R},\ i = 1, \dots, n \}$$

$V = \{v_1, v_2, \dots v_n\}$

$\vec{w}_1 \in \mathrm{Span}(V)$ och $\vec{w}_2 \in \mathrm{Span}(V) \Longrightarrow$

$w_1 + w_2 \in \mathrm{Span}(V)$

$w_1 = \alpha_1 v_1 + \dots + \alpha_n v_n$

$\underline{w_2 = \beta_1 v_1 + \dots + \beta_n v_n}$

$(w_1 + w_2) = (\alpha_1 + \beta_1) v_1 + \dots + (\alpha_n + \beta_n) v_n$

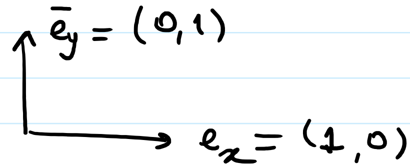
Ex: $\quad V = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$$\mathrm{Span}(V) = \left\{ w = \alpha \underbrace{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}_{v_1} + \beta \underbrace{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}}_{v_2} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$$

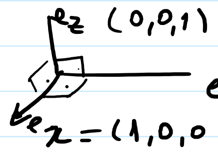
$$w = \begin{pmatrix}\alpha\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\\beta\\0\end{pmatrix} = \begin{pmatrix}\alpha\\0\\\beta\\0\end{pmatrix}$$

Span(V) är ett <u>plan</u> i $\underline{\mathbb{R}^4}$

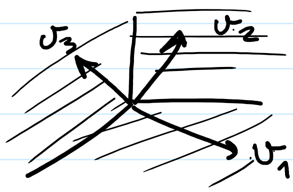
<u>Bas</u>: $\underbrace{\mathbb{R}^n}$



$$\bar{e}_1 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} \quad \bar{e}_2 = \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix} \quad \cdots \quad \bar{e}_n = \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}$$



$$B = \{ \bar{e}_1, \bar{e}_2, \dots, \bar{e}_n \}$$



$$\vec{e}_i = \begin{pmatrix} x_k \end{pmatrix}, \quad \begin{matrix} x_k = 0 & k \neq i \\ x_k = 1 & k = i. \end{matrix}$$

$$T: V \longrightarrow W \qquad V = \mathbb{R}^n, \; W = \mathbb{R}^m$$

$$u \longrightarrow T(u)$$

T = linjär transformation
avbildning
funktion

$\Updownarrow$

$M_{n \times m}$

<u>Properties</u> of Matrises

$$A_{n\times n} = \left[a_{ij} \right] \qquad \begin{matrix} i = 1, \dots n \\ j = 1, \dots n \end{matrix}$$

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \dots & & a_{2n} \\ \vdots & & & \\ a_{m1} & \dots & & a_{mn} \end{bmatrix} = \left[a_{ij} \right] \qquad \begin{matrix} i = 1, \dots m \text{ (rad)} \\ j = 1 \dots, n \text{ (kolumn)} \end{matrix}$$

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$v \Longrightarrow T_A(v) = A \cdot v = A_{m\times n} v_{n\times 1}$$

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_{n\times 1}$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}_{m\times 1}$$

$$A_{m\times n} + B_{m\times n} = \left[a_{ij} + b_{ij} \right] \qquad \begin{matrix} i = 1, \dots, m \\ j = 1, \dots, n \end{matrix}$$

$$\underset{m\times n}{A} \cdot \underset{n\times k}{B} = \underbrace{\begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}}_{A} \begin{bmatrix} b_1 & \cdots & b_k \end{bmatrix} \circledast$$

$$= \Big[\underbrace{[A]_{m\times n} \cdot b_{1\,n\times 1}}_{m\times 1} \quad \underbrace{A_{m\times n} b_{n\times 1}}_{m\times 1} \ [A] b_2 \quad \cdots \quad [A] b_k \Big]$$

Sidan 132

$$v = (v_1\ v_2\ v_3)_{1\times 3}$$

$$w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}_{3\times 1}$$

$$\underset{1\times 3}{(v_1\ v_2\ v_3)} \begin{pmatrix} w_1 \\ w_2 \\ w_2 \end{pmatrix}_{3\times 1} = (\underbrace{v_1 w_1 + v_2 w_2 + v_3 w_3}_{\in \mathbb{R}})$$

$$v \cdot w \in \mathbb{R}$$

$$w_{3\times 1} \cdot v_{1\times 3} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \cdot (v_1\ v_2\ v_3) = \underbrace{\begin{pmatrix} w_1 v_1 & w_1 v_2 & w_1 v_3 \\ w_2 v_1 & w_2 v_2 & w_2 v_3 \\ w_3 v_1 & w_3 v_2 & w_3 v_3 \end{pmatrix}}$$