F_5.1=9_alice

den 8 februari 2021 06:58

Mål
- Kap 4 $V = \mathbb{R}^n$ $W = \mathbb{R}^m$ $A_{n\times n}$, $A_{m\times n}$
- Kap 5 : Linjära Ekvationssystem
  Gausseliminationen + Lösning

$V = \mathbb{R}^n$

$$A_{n\times n} = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{bmatrix} \qquad a_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} \qquad i = 1, \dots, n$$

$$A^t_{n\times n} = \begin{bmatrix} - a_1 - \\ - a_2 - \\ \vdots \\ - a_n - \end{bmatrix}$$

$$A_{m\times n} \longrightarrow A^t_{n\times m}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad \text{vektor} \quad v_{n\times 1}$$

$$v^t = [v_1\ v_2 \dots v_n]_{1\times n} \qquad (v = (v_1, \dots, v_n))$$

$$u \bullet v = \sum_{k=1}^{n} u_k \cdot v_k \qquad u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\underbrace{u^t_{1\times n}\, v_{n\times 1}}_{} = [u_1\ u_2 \dots u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \underbrace{u_1v_1 + u_2v_2 + \dots + u_nv_n}_{\sum_{k=1}^{n} u_k v_k \,=\, u \bullet v}$$

$$u\, v^t = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}_{n\times 1} [v_1 \dots v_n]_{1\times n}$$

$$= \begin{bmatrix} u_1v_1 & u_1v_2 & \dots & u_1v_n \\ u_2v_1 & u_2v_2 & \dots & u_2v_n \\ \vdots & & & \\ u_nv_1 & u_nv_2 & \dots & u_nv_n \end{bmatrix}_{n\times n} = \begin{bmatrix} u_1[v]^t \\ u_2[v]^t \\ \vdots \\ u_n[v]^t \end{bmatrix}_{n\times n} \quad (*)$$

$$A_{2\times 2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \tilde{A} = \begin{bmatrix} ka & kb \\ \alpha c & \alpha d \end{bmatrix}$$

$\det \tilde{A} = k\alpha d - k bc = k\alpha(ad - bc) = k\alpha \det A$

$A_{2\times 2} = \begin{bmatrix} a & b \\ a & b \end{bmatrix} \rightarrow \det A = ab - ab = 0$

1) $(A_{n\times n} . B_{n\times n})^t = B^t . A^t$ page 132

2) $\det(A) = \det(A^t)$

Om det finns $B_{n\times n}$ så at $B.A = A.B = I_{n\times n} = I_n$

$I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{n\times n}$

A kallas inverterbar och $B = A^{-1}$

$\det A \neq 0 \Leftrightarrow$ A är inverterbar.
OM och endast om

$\det(A.B) = \det A . \det B$

$\underbrace{\det(A.A^{-1})}_{\det(I) = 1} = \det A . \det A^{-1}$

$\det A . \det A^{-1} = 1 \Longrightarrow$

$$\boxed{\det A^{-1} = \frac{1}{\det A}}$$

Prop 4.27 $A_1 A_2 \cdots A_r$ r stycken Inverterbara $n\times n$ matrisen

Då gäller att

$$(A_1 A_2 \cdots A_r)^{-1} = A_r^{-1} A_{r-1}^{-1} \cdots A_2^{-1} . A_1^{-1}$$

Bevis. Induktion: Egenskap (sats)

? är det bra för $r = 2$?

Induktions Hypotes: $r < K$

$r < K$ ✓ $\Longrightarrow$ $k+1$ är också bra.

1) $r = 2$ $\quad (A_1 \cdot A_2)^{-1} \stackrel{?}{=} A_2^{-1} \cdot A_1^{-1}$ Ja vill bevisa det.

$$\boxed{(A_1 A_2) \circ (A_2^{-1} \cdot A_1^{-1}) =}$$

$$A_1 \, (\underbrace{A_2 \cdot A_2^{-1}}_{I_n}) \, A_1^{-1}$$

$$(A_1 \cdot I_n) \, A_1^{-1} \Rightarrow A_1 \cdot A_1^{-1} = I$$

Då $(A_2^{-1}) \cdot (A_1^{-1})$ är inversa matris av $(A_1 A_2)$

Induktion

Hypotes : $(A_1 \cdot A_2 \ldots A_k)^{-1} = A_k^{-1} \cdot A_{k-1}^{-1} \cdots A_2^{-1} \cdot A_1^{-1}$ gäller

$$\boxed{r = k \Longrightarrow r = k+1}$$

$$(A_1 \cdot A_2 .. A_k \cdot A_{k+1})^{-1} =$$

$$\big((\underbrace{A_1 A_2 \ldots A_k}_{r=1}) \underbrace{A_{k+1}}_{r=2}\big)^{-1} \underset{r=2}{=} (A_{k+1})^{-1} \cdot \underbrace{(A_1 \ldots A_k)^{-1}}_{\text{Hypotes}} =$$

$$= A_{k+1}^{-1} \cdot (A_k^{-1} \cdot A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1})$$

Enligt M. Induktion satsen gäller för alla $r \in \mathbb{N}$.

$V = \mathbb{R}^n$ $\quad$ L: linje $\quad X = P_0 + t\,v$, t parameter $\in \mathbb{R}$

v = riktnings vektor

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad P_0 = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

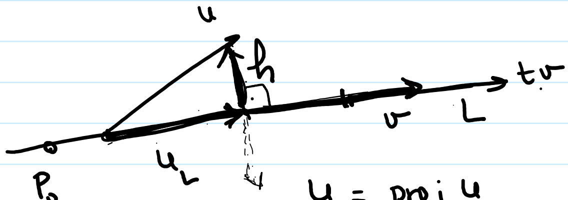
$V = \mathbb{R}^n$ $\quad \Pi$: planet $\quad X = P_0 + t\,v + s\,w$

v, w är riktnings vektor $\quad t, s \in \mathbb{R}$ fria parameter.

$$f : V \longrightarrow V \qquad V = \mathbb{R}^n \qquad g : V \longrightarrow W$$

$$u \longrightarrow f(u) \qquad W = \mathbb{R}^m \qquad v \longrightarrow g(v)$$

f = projektion mot linje L



$$u_L = \text{proj}_v\, u$$

$$\boxed{u = u_L + h}$$

$$h \perp v \Leftrightarrow h \neq \alpha v$$

$$u_L \,//\, v \Leftrightarrow u_L = cv$$

$$u = cv + h$$

$$u \cdot v = c\, v \cdot v + \underbrace{h \cdot v}_{=0} \qquad h \perp v$$

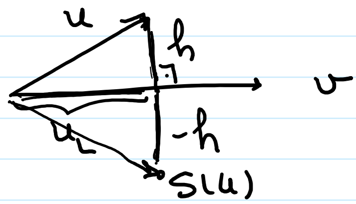
$$\Rightarrow c = \frac{u \cdot v}{v \cdot v}$$

$$\Rightarrow u_L = \left(\frac{u \cdot v}{v \cdot v}\right) v \quad \Rightarrow u_L = \frac{v^t\, u}{\| v \|^2}\, v$$

$$u_L = \underbrace{\left(\frac{v^t \cdot u}{\| v \|^2}\right)}_{\alpha} v_{n\times 1} = \underbrace{v}\left(\frac{v^t \cdot u}{\| v \|^2}\right) = \frac{1}{\| v \|^2} \underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \begin{bmatrix} v_1 \cdots v_n \end{bmatrix}}_{\text{Matris}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$f(u) = u_L = \underbrace{\frac{1}{\| v \|^2}(v \cdot v^t)}\, u = \frac{1}{\| v \|^2} \underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \begin{bmatrix} v_1 \ldots v_n \end{bmatrix}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Speglingen



$$\boxed{u = u_L + h} \quad \Rightarrow h = u - u_L$$

$$S(u) = u_L - h$$

$$u_L - (u - u_L) =$$

$$u = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$S(u) = \underbrace{2u_L} - u$$

$$= \underbrace{\frac{2}{\|v\|^2} v \cdot v^t} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - I_n \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \underbrace{\left(\frac{2}{\|v\|^2} v \cdot v^t - I \right)}_{S} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$f: \underset{\mathbb{R}^n}{V} \longrightarrow \underset{\mathbb{R}^m}{W}, \quad V = \mathbb{R}^n \quad W = \mathbb{R}^m, \quad f \text{ linjär}$$

$$u \longmapsto f(u) = A \cdot u = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \underbrace{\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}}_{?} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}$$

given

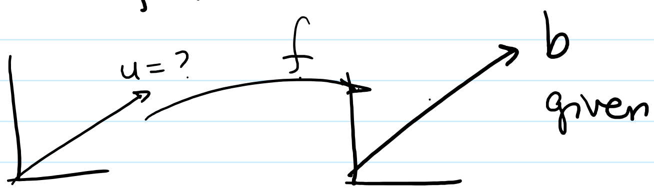
$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

$$f(e_1) \quad f(e_2) \quad \cdot \quad f(e_n)$$

$$B = \{e_1, e_2 \dots e_n\}$$

$$u = u_1 e_1 + u_2 e_2 + \dots + u_n e_n$$

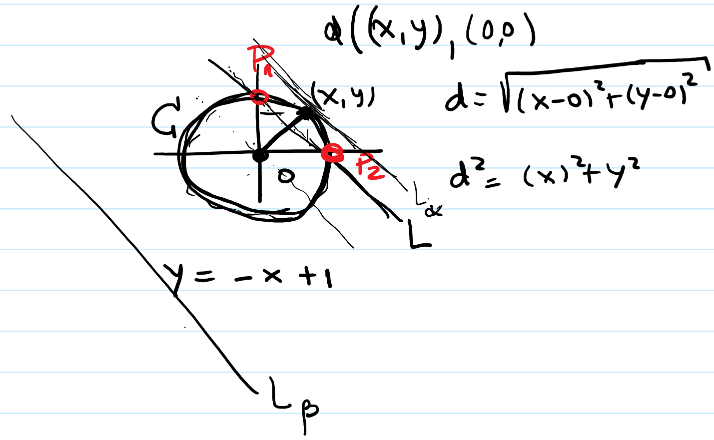
$$f(u) = u_1 f(e_1) + \dots + u_n f(e_n)$$



Kapitel 5 Linjära Ekvationssystem

1) Ej linjär Ekvationssystem

$$\begin{matrix} C \\ L \end{matrix} \left\{ \begin{aligned} x^2 + y^2 &= 1 \\ x + y &= 1 \quad \alpha \quad \beta \end{aligned} \right.$$



Lösning.

$$L: \boxed{y = 1 - x}$$

$$C: \quad x^2 + (1-x)^2 = 1$$

$$x^2 + \cancel{1} - 2x + x^2 = \cancel{1}$$

$$\underbrace{x^2 + \cancel{1} - 2x + x^2} = \cancel{1}$$

$$\cancel{2}x^2 - \cancel{2}x = 0 \Rightarrow x(x-1) = 0 \Rightarrow \begin{cases} x = 0 \\ x = 1 \end{cases} \text{ eller}$$

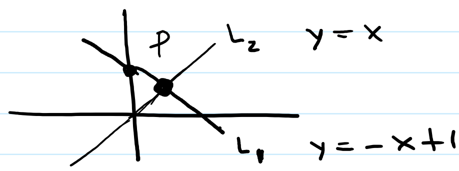
$$x = 0 \Rightarrow y = 1 - 0 \Rightarrow y = 1 \Rightarrow P_1 = (0, 1)$$

$$x = 1 \Rightarrow y = 1 - 1 \Rightarrow y = 0 \Rightarrow P_2 = (1, 0)$$

Linjär System

$V = \mathbb{R}^2$

$$\begin{array}{l} L_1 \\ L_2 \end{array} \begin{cases} x + y = 1 \\ x - y = 0 \end{cases}$$



Lösning.

$$\begin{cases} x + y = 1 \\ x - y = 0 \end{cases}$$

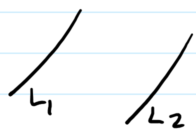
$$2x \; / = 1 \Rightarrow \boxed{x = \frac{1}{2}}$$

$$\frac{1}{2} - y = 0 \Rightarrow \boxed{y = 1/2}$$

$P = (1/2, 1/2)$ lösning

$V = \mathbb{R}^2$

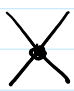
$L_1 // L_2$ men $L_1 \neq L_2$



$L_1 \cap L_2 = \emptyset$ Det finns inge lösning

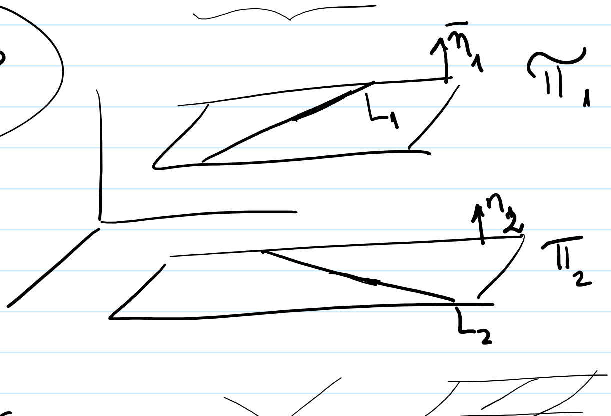
$L_1 // L_2$ och $L_1 = L_2$

$L_1 \cap L_2 = L_1 = L_2$ ∞ Oändliga lösning

L_1  L_2

L_1 not parallell med L_2

$V = \mathbb{R}^3$

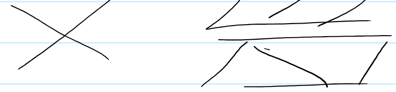


$\Pi_1 // \Pi_2$

$\Pi_1 \cap \Pi_2 = \emptyset$

$\bar{n}_1 // \bar{n}_2 \quad n_1 = \alpha n_2$

Linjär System



1) $\{ ax = b$ om $a \neq 0$ då $x = \frac{b}{a}$

1) $\{ ax + by = c$ om $a \neq 0$ då $\qquad V = \mathbb{R}^2$

$$ax = c - by \Rightarrow \boxed{x = \frac{c}{a} - \frac{b}{a} y}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c/a - b/a\, y \\ y \end{pmatrix} = \begin{pmatrix} c/a \\ 0 \end{pmatrix} + \cancel{y}\, t \begin{pmatrix} -b/a \\ 1 \end{pmatrix} \qquad t \in \mathbb{R}$$

fri parameter

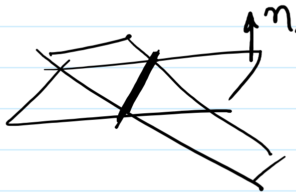
$$t = 0 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c/a \\ 0 \end{pmatrix}$$

$$t = 1 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c/a \\ 0 \end{pmatrix} + 1 \begin{pmatrix} -b/a \\ 1 \end{pmatrix} = \begin{pmatrix} c/a - b/a \\ 1 \end{pmatrix}$$

$$t = a \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c/a \\ 0 \end{pmatrix} + a \begin{pmatrix} -b/a \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} c/a \\ 0 \end{pmatrix} + \begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} c/a - b \\ a \end{pmatrix}$$

$$\begin{aligned} \pi_1 \;&\{ 1x + 1y + 1z = 1 \\ \pi_2 \;&\; x + y - z = 1 \end{aligned}$$



$n_1 = (1,1,1)$

$n_2 = (1,1,-1)$

$$\tfrac{1}{2}(2x + 2y \quad = 2)$$

$$x + y = 1 \qquad y = -x + 1$$

$$\cancel{x} + (-\cancel{x} + 1) + z = 1$$

$$\boxed{z = 0}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 0 \\ -x + 1 \\ 0 + 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \cancel{x}\, t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}$$

Prop 5.2

$\begin{cases} A_{n \times n} \text{ inverterbar } \mid (A \text{ är kvadrat}) \quad (\text{det finns } A^{-1}) \\ Ax = b \end{cases}$

$\boxed{\text{Då lösning } x = A^{-1} \cdot b}$

Bevis

$$\boxed{Ax = b}$$

Vi vet vem är A^{-1} (A är inverterbar)

Då $\underbrace{A^{-1} \cdot A}\, x = A^{-1} \cdot b$

$$I \cdot x = A^{-1} b$$

$$\boxed{x = A^{-1} \cdot b}$$

Frågor: • $A^{-1}_{n \times n}$? Hur kan man räkna $A^{-1}_{n \times n}$ när $n \geq 3$.

∘ Hur kan man lösa systemet när är ej kvadrat

$A_{m \times n}$ $\quad m \neq n$.

System. $\begin{cases} 1x + 1y = 1 \\ -x + y = 1 \end{cases} \iff \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{A_{2\times 2}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A \cdot X = B$$

A inverterbar $\iff \det A \neq 0$ $\qquad [A \mid B]$

$\det A = 1 - (-1) = 2$

$S_1 \begin{cases} 1x + y = 1 \\ 0 + 2y = 2 \end{cases} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad \frac{1}{2}$ $\quad \frac{1}{2}$

$y = \frac{2}{2} = 1$ $\qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$1x + 1 = 1$

$x = 0$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

$\underbrace{}_{A}$ $\underbrace{}_{b}$

$S_1 \begin{cases} 0 + 2y = 2 \\ 1x + y = 1 \end{cases}$

Total matris $(a_1 \ldots a_n\ b)$

Ex.

$$\begin{cases} 3x_1 + 6x_2 + 9x_3 = 36 \\ 2x_1 + 7x_2 - 2x_3 = 38 \\ 4x_1 + 11x_2 + 24x_3 = 102 \end{cases}$$

$$\underbrace{\begin{pmatrix} 3 & 6 & 9 \\ 2 & 7 & -2 \\ 4 & 11 & 24 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 36 \\ 38 \\ 102 \end{pmatrix}}_{B}$$

$$\begin{array}{c} R_1 \\ R_2 \end{array} \left[\begin{array}{ccc|c} 3 & 6 & 9 & 36 \\ 2 & 7 & -2 & 38 \end{array}\right] \qquad \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & b_2 \end{array}\right]$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 3 & 6 & 9 & 36 \\ 2 & 7 & -2 & 38 \\ 4 & 11 & 24 & 102 \end{array}\right] \xrightarrow{\text{Elementära Operationer}} \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & b_2 \\ 0 & 0 & a_{33} & b_3 \end{array}\right]$$

$$\begin{cases} R_i \leftrightarrow R_j & \text{byt ar plats raderna} \\ \alpha R_i \\ \alpha R_i + R_j \end{cases}$$

$$\Downarrow$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{33}x_3 = b_3$$

om $a_{33} \neq 0$