F_5.2=10_alice

den 18 februari 2021 07:22

Mål → • $Ax=b$ → Gausselimination → $\boxed{\text{Trappstegform}}$

→ Lösning $x = x_p + x_H$ med $Ax_H = 0$

$A_{m\times n}$ $m \leq n$

$A_{n\times n}$ inverterbar då $Ax=b \Rightarrow \underbrace{x = A^{-1}b}$

• $Ax=b$ med $m > n$ Överbestämd system

• $\det A_{n\times n}$ $n \geq 2$

$\underbrace{\text{Elementära Operation}}$ • $E_{i\leftrightarrow j}$ byta Ekvation i med E_j

$E_{\alpha,i}$ αE_i

$E_j \leftarrow E_{\alpha,i} + E_j$

$$Ax=b \quad \begin{matrix} E_1 \\ \vdots \\ E_n \end{matrix} \left\{ \begin{array}{l} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{n1}x_1 + \dots + a_{nn}x_n = b_n \end{array} \right. \longleftrightarrow T = (A \mid \tilde{b})$$

Prop 5.9
sidan 153

$Ax=b$ → $\underbrace{T = (A \mid b)}$ $\Longrightarrow$ $S = (\tilde{A} \mid \tilde{b})$

Gausselimination
Elementära operation.

T och S är ekvivalenta: d.v.s

T och S har samma lösning.

Exempel:

$$\left\{ \begin{array}{l} 1x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 4x_2 + 3x_3 = 3 \\ 3x_1 + 6x_2 + 6x_3 = 4 \end{array} \right. \longleftrightarrow x_1 \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{v_1} + x_2 \underbrace{\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}}_{v_2} + x_3 \underbrace{\begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix}}_{v_3} = \underbrace{\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}}_{b}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 3 & 3 \\ 3 & 6 & 6 & 4 \end{array} \right) \begin{array}{l} \text{pivot rad} \\ R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1 \end{array}$$

$$\begin{bmatrix} f(e_1) & f(e_2) & f(e_3) \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

b

$$\begin{array}{c} \\ R_2 \rightarrow \\ \\ \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & -3 & 1 \end{array} \right) \begin{array}{l} \\ \text{pivot rad} \\ R_3 \leftarrow R_3 - R_2 \end{array}$$

$$R_2 \to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & -3 & 1 \end{array}\right) \begin{array}{l} \text{pivot rad} \\ R_3 \leftarrow R_3 - R_2 \end{array}$$

$$\begin{array}{l} \text{Pivot} \to \\ \text{Pivot} \to \\ \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \longleftrightarrow \begin{cases} 1x_1 + 2x_2 + 3x_3 = 1 \\ \quad / \quad / \quad -3x_3 = 1 \\ \quad / \quad / \quad / \quad / \end{cases} \Longrightarrow$$

↑ fria kolumn

$$x_3 = -\tfrac{1}{3}$$

$$x_1 = 1 - 2x_2 - 3x_3$$

$$x_1 = 1 - 2x_2 + \cancel{3}\left(+\tfrac{1}{\cancel{3}}\right) =$$

$$x_1 = 2 - 2x_2$$

$$x_1 = 2 - 2x_2$$

$$x_2 = 0 + 1x_2$$

$$x_3 = -\tfrac{1}{3} + 0x_2$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -\tfrac{1}{3} \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}$$

Interpretation 1

$$\left.\begin{array}{ll} E_1 : & \Pi_1 \\ E_2 & \Pi_2 \\ E & \Pi_3 \end{array}\right\} \Pi_1 \cap \Pi_2 \cap \Pi_3 = \text{linje}$$

Anmärkning: Vi kan fortsätta med Gauss elimination till alla pivot är 1 och ensam i sina kolumner.

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 1 \\ 0 & 0 & \boxed{-3} & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) R_2 \leftarrow -\tfrac{1}{3} R_2$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 1 \\ 0 & 0 & \boxed{1} & -\tfrac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} R_1 \leftarrow R_1 - 3R_2 \\ \text{pivot} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -\tfrac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow$$

$$\begin{cases} 1x_1 + 2x_2 + 0x_3 = 2 \\ x_2 = x_2 \end{cases}$$

$$\begin{cases} 1x_1 + 2x_2 + 0\,x_3 = 2 \\ x_2 = x_2 \\ 0 + 0 + 1x_3 = -1/3 \end{cases}$$

$$1x_1 = 2 - 2x_2$$

$$x_2 = 0 + x_2$$

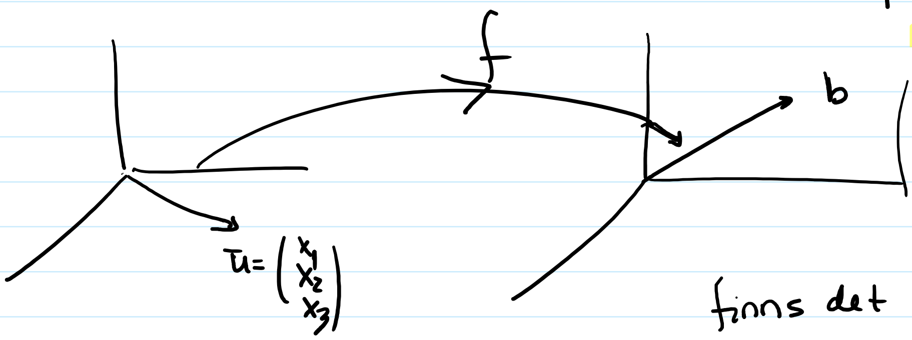
$$x_3 = -1/3 + 0$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1/3 \end{pmatrix} + \cancel{x_2}\,t \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \; t \in \mathbb{R}$$

Interpretation 2:

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$u \longrightarrow f(u) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 3 \\ 3 & 6 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

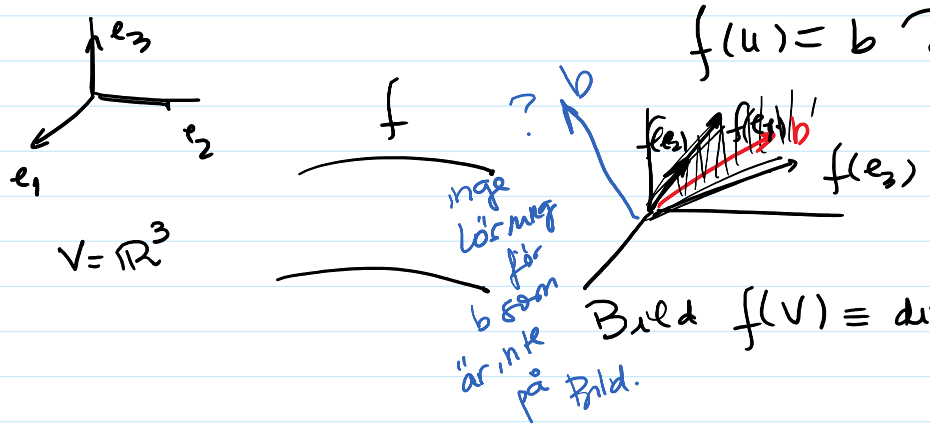


$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$u = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

finns det några vektor $u = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ i definitionsmängd så att

$$f(u) = b \,?$$



$V = \mathbb{R}^3$

Bild $f(V) \equiv$ dim $= \text{Span}\{f(e_1), f(e_3)\}$

§ 5.4 Sidan.

Satsen 5.20 $\qquad Ax = b \qquad \boxed{A_{n\times n}} \qquad T = (A|b)$

Trappstegform skapad med T

Detta är ekvivalenta.

1) Det finns unik lösning för alla b

2) Matrisen T saknar fria kolumner
   (alla kolumner har en pivot element)

(1, 2) falsk — då finns det ingen lösningar $\left(\begin{array}{ccc|c} a_{11} & & & \\ 0 & 0 & 0 & \tilde{b}_n \end{array}\right)$?

eller

då finns det oändligt många lösningar $\left(\begin{array}{ccc|c} a_{11} & & & \\ 0 & 0 & 0 & 0 \end{array}\right)$

S. 5.20 : $A_{n\times n}$ är inverterbar

$$A \cdot B = I \qquad \boxed{B = A^{-1}}$$

$A \cdot B = B \cdot A = I$

$B = A^{-1}$

$A = B^{-1}$

$$\underbrace{\begin{pmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{pmatrix}}_{} = \begin{pmatrix} 1 & 0 & & 0 \\ | & 1 & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}$$

$$\boxed{A x_1 = e_1 \qquad A x_2 = e_2 \quad \cdots \quad A x_n = e_n}$$

Sidan 164.

$$A = \begin{pmatrix} 3 & 6 & 9 \\ 0 & 2 & -8 \\ 0 & 0 & 5 \end{pmatrix} \qquad A x_1 = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad A x_2 = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$A x_3 = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T = \left(\begin{array}{ccc|ccc} 3 & 6 & 9 & 1 & 0 & 0 \\ 0 & 2 & -8 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right) \quad \tfrac{1}{3} R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & \tfrac{1}{3} & 0 & 0 \\ 0 & 2 & -8 & 0 & \tfrac{1}{2} & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 \leftarrow R_1 - R_2 \\ R_2 \leftarrow \tfrac{1}{2} R_2 \\ R_3 \leftarrow \tfrac{1}{5} R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 11 & \tfrac{1}{3} & -1 & 0 \\ 0 & 1 & -4 & 0 & \tfrac{1}{2} & 0 \\ & & 1 & 0 & 0 & \tfrac{1}{5} \end{array}\right) \quad \begin{array}{l} R_1 \leftarrow R_1 - 11 R_3 \\ R_2 \leftarrow R_2 + 4 R_3 \\ \text{Pivot} \end{array}$$

$| 0 \quad 0 \quad 1 \; | \; 0 \quad 0 \quad 1/5 \, |$ Pivot

$$\left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \;\middle|\; \underbrace{\begin{matrix} 1/3 & -11 & -11/5 \\ 0 & 1/2 & 4/5 \\ 0 & 0 & 1/5 \end{matrix}}_{A^{-1} = B}\right)$$

Testa

$$\begin{pmatrix} 3 & 6 & 9 \\ 0 & 2 & -8 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1/3 & -11 & -11/5 \\ 0 & 1/2 & 4/5 \\ 0 & 0 & 1/5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$A X = 0$ Homogenea system

$A x = b$

Prop 5.31 sidan 171

$A x = 0$
$X = 0$ unik lösning $\Longleftrightarrow$ Reducering av T till trappstegform har inga fri kolumn.

$A x = 0$
X har oändligt många lösning $\Longleftrightarrow$ Reducering av T till trappstegform har minst en fri kolumn.

Exempel.

$$\begin{cases} 2x_1 + 3x_2 - x_3 + x_4 = 0 \\ -x_1 + 4x_2 - 2x_3 - 2x_4 = 0 \\ 3x_1 - x_2 + x_3 + 3x_4 = 0 \end{cases}$$

$T: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$

$\bar{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longrightarrow T(\bar{u}) = A \cdot u$

$V = \mathbb{R}^4$

5.32

Lösning: $(T \mid 0)$

Lösning: $(T \mid 0)$

$$\left(\begin{array}{cccc|c} 2 & 3 & -1 & 1 & 0 \\ -1 & 4 & -2 & -2 & 0 \\ 3 & -1 & 1 & 3 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 2/11 & 10/11 & 0 \\ 0 & 1 & -5/11 & -3/11 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right)$$

oändligt lösningar

$f(e_1)$ $f(e_2)$ $f(e_3)$ $f(e_4)$ fri fri

$$\begin{cases} \boxed{1x_1} + 0x_2 + \frac{2}{11}x_3 + \frac{10}{11}x_4 = 0 \\ \quad \boxed{1x_2} - \frac{5}{11}x_3 - \frac{3}{11}x_4 = 0 \end{cases}$$

$$x_1 = -\frac{2}{11}x_3 - \frac{10}{11}x_4$$

$$x_2 = +\frac{5}{11}x_3 + \frac{3}{11}x_4$$

$$x_3 = t$$

$$x_4 = s$$

Lösning: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t\begin{pmatrix} -2/11 \\ 5/11 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -10/11 \\ 3/11 \\ 0 \\ 1 \end{pmatrix}$ $s, t \in \mathbb{R}$

Sats. 5.33 $A_{m\times n} X_{n\times 1} = b_{m\times 1}$

Lösning av systemet skrivas som.

$$\boxed{X = X_p + X_H}$$

X_H lösning för Hom. syst $AX_H = 0$

Bevis: x är lösning

$\Rightarrow$ $A(X_p + X_H) = AX_p + AX_H = b + 0 = \underline{b}$

$\Leftarrow$ X_ℓ om vi har en andra lösning particular

$x = x_p + x_H$? $x_\ell - x_p$?

$A(\underbrace{x_\ell - x_p}) = \underbrace{Ax_\ell} - \underbrace{Ax_p} = b - b = \underline{0}$

$\Longrightarrow$ $x_\ell - x_p =$ lösning till Homg.

$x_\ell - x_p = x_H \Longrightarrow \boxed{x_\ell = x_p + x_H}$

förra ex

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & \cancel{1} \\ 0 & 0 & 1 & \cancel{-1/3} \\ 0 & 0 & 0 & \cancel{0} \end{array}\right)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \cancel{\begin{pmatrix} 2 \\ 0 \\ -1/3 \end{pmatrix}} + \underbrace{t\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}_{x_H}$$

$t = 0$

$$\underbrace{\begin{pmatrix} 2 \\ 0 \\ -1/3 \end{pmatrix}}_{x_p}$$

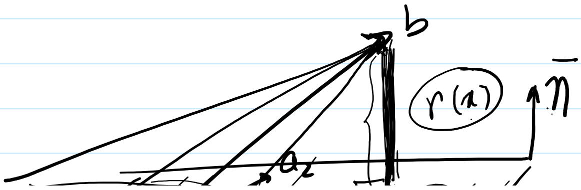
$A_{m \times n} x = b \qquad m > n$

$$\begin{cases} x_1 + x_2 = 1 \\ 3x_1 - x_2 = 3 \\ 1x_1 - 5x_2 = 7 \end{cases} \leftarrow$$

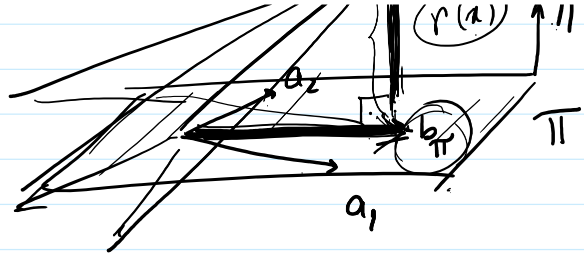
Ny Metod : Minska Kvadratlösning sidan 173

$Ax = b$

$$\begin{pmatrix} | & | \\ a_1 & a_2 \\ | & | \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$



$$\begin{pmatrix} | & | \\ a_1 & a_2 \\ | & | \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_3 \end{pmatrix}$$



$$x_1\begin{pmatrix} | \\ a_1 \\ | \end{pmatrix} + x_2\begin{pmatrix} | \\ a_2 \\ | \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}$$

$$\text{Span}\{a_1, a_2\} = \{x_1a_1 + x_2a_2\} = \{Ax, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_i \in \mathbb{R}\}$$

$$r = \|b - b_\pi\| \leq \|Ax - b\|$$

original

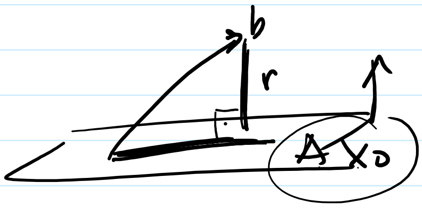
$$\|b_\pi - b\| = \|Ax_0 - b\|$$

vi vill hitta.

Hur bestämmer vi $b_\pi = Ax_0$

$x_0 =$ är bästa lösning att vi kan ha.

1) $r = \underbrace{Ax_0}_{b_\pi} - b$



$r \parallel \eta_\pi \qquad r \perp a_1 \qquad r \perp a_2 \qquad (r \perp \pi)$

$$a_1 \cdot r = a_1^t \cdot r = 0$$

$$a_2 \cdot r = a_2^t \cdot r = 0$$

$$\Longrightarrow \boxed{A^t \cdot r = 0}$$

2) $A^t \cdot r = 0$

$$A^t(Ax_0 - b) = 0$$

$$A^tAx_0 - A^tb = 0$$

$$\boxed{A^tAx_0 = A^tb}$$

Kvadrat system.

$$\underbrace{\underbrace{A^t_{n\times m}\, A_{m\times n}}_{n\times n}}\cdot \underbrace{\begin{pmatrix} x_{01} \\ \vdots \\ x_{0n} \end{pmatrix}}_{n\times 1} = \underbrace{A^t_{n\times m} \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}_{m\times 1}}_{n\times 1}$$

$$\boxed{A^t A x = A^t b}$$

$$\Downarrow$$

$$x_0 \in \text{Span}\{a_1, a_n \ldots a_n\}$$