F_7.1=13

den 1 mars 2021 07:24

Basbyte $A_E = G\, A_G\, G^{-1}$ G= base byte matris

$f: \mathbb{R}^n \to \mathbb{R}^n$

$u \to f(u) = Au$

§7.4 ON-bas

Alice Kozakevicius at 2021-03-01 08:04

Def: $G = (g_1\, g_2 \dots g_n)$ är ortonormal bas (ON-bas)

om: $\begin{cases} g_i \cdot g_j = 0 & i \neq j \\ \|g_i\| = 1 & (g_i \cdot g_i) = 1 \end{cases}$

Def 7.32 $f_A: \mathbb{R}^n \to \mathbb{R}^n$ linjär avb.

$u \to f_A(u) = A\cdot u$

f_A är isometrisk om f_A bevarar längder

d.v.s $\|f_A(u)\| = \|A(u)\| = \|u\|$

Motsvarande matris A kallas ON-Matris

Prop. 7.34 $f_A: \mathbb{R}^n \to \mathbb{R}^n$ en isometrisk linjär avbildning

Då bevarar f_A skalär produkter

$(f_A(u)) \cdot (f_A(v)) = Au \cdot Av = u \cdot v$, for all vektors $u, v \in \mathbb{R}^n$

Bevis:

- $\|u\|^2 = u \cdot u$

$\|u+v\|^2 = (u+v)\cdot(u+v) = u\cdot u + u\cdot v + v\cdot u + v \cdot v =$

$\|u+v\|^2 = \|u\|^2 + 2u\cdot v + \|v\|^2$

$$\boxed{\frac{1}{2}\left(\|u+v\|^2 - \|u\|^2 - \|v\|^2\right) = u \cdot v}$$

$u = Ax$

$v = Ay$

$u+v = Ax + Ay = A(x+y)$

A isometrisk

$\Rightarrow (Ax)\cdot(Ay) = \frac{1}{2}\left(\|A(x+y)\|^2 - \|Ax\|^2 - \|Ay\|^2\right) =$

$= \frac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right) = x \cdot y$

Satsen 7.35 $A_{n\times n} = \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix}$ ON-matris $\Leftrightarrow$ $(a_1\, a_2 \dots a_n)$ är ON-bas för $\mathbb{R}^n$

$\boxed{\Rightarrow}$ $A_{n\times n}$ = ON-Matrix

$\begin{cases} \|Ax\| = \|x\| \\ Ax \cdot Ay = x \cdot y \end{cases}$ for all $x, y \in \mathbb{R}$

In particular if $x = e_i$ $y = e_j$ $E = (e_1 \dots e_n)$ standard base

$i = 1, \dots n$

$j = 1 \dots n$

$\|Ae_i\| = \|e_i\| = 1$ $\quad Ae_i = a_i$

$\underbrace{Ae_i}_{a_i} \cdot \underbrace{Ae_j}_{a_j} = e_i \cdot e_j = 1 \Longrightarrow (a_1\, a_2 \dots a_n)$ is an ON-bas

$\underbrace{Ae_i}_{a_i} \cdot \underbrace{Ae_j}_{a_j} = e_i \cdot e_j = 1 \implies (a_1\, a_2 \ldots a_n)$ is an ON-bas

Ex $(a_1 \ldots a_n)$ ON-base

$u \in \mathbb{R}^n \Rightarrow u = \alpha_1 a_1 + \ldots + \alpha_n a_n = A u_A$

$\|Au\|^2 = Au_A \cdot Au_A = (\sum_{i=1}^n \alpha_i a_i) \cdot (\sum_{j=1}^n \alpha_i a_i) =$

$\sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \underbrace{(a_i)\cdot(a_j)}_{1 \text{ if } i=j} = \sum_{i=1}^n \alpha_i^2 = \|u\|^2$

Satsen 7.37 $A_{n\times n}$ ON-matris $\iff \boxed{A^{-1} = A^t}$

Bevis: $A^{-1}_{n\times n} A_{n\times n} = I_n$
$\quad\quad = A \cdot A^{-1}_{n\times n}$

$A = \begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{pmatrix} \qquad A^t = \begin{pmatrix} - a_1^t - \\ - a_2^t - \\ \vdots \\ a_n^t \end{pmatrix}$

$A^t \cdot A = \begin{pmatrix} - a_1^t - \\ - a_2^t - \\ - a_n^t - \end{pmatrix} \begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} a_1^t \cdot a_1 & a_1^t a_2 & \cdots & a_1^t a_n \\ & \ddots & & \\ a_n^t a_1 & & & a_n^t a_n \end{pmatrix}$

$= \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = I_n$

7.35 $f: \mathbb{R}^n \to \mathbb{R}^n$ A_G = matris $G = (g_1 \ldots g_n)$

$A_E = G \cdot A_G\, G^{-1}$

If G is ON-bas

$\boxed{A_E = G \cdot A_G \cdot G^t}$

Kapitel 8 Egenvärde & Egenvektorer

Def $A_{n\times n}$

OM det finns $\lambda \in \mathbb{R}$, $v \neq 0$ så att $\boxed{Av = \lambda v}$ då kallas

λ = egenvärde till A

v = egenvektor till A

Obs $Av = \lambda v \iff Av \,//\, \lambda v$ ($\{Av, \lambda v\}$ är L Beroende

Exampels

1) $A_{n\times n} = s I_n = \begin{pmatrix} s & 0 & \cdots & 0 \\ 0 & s & 0 & \cdots \\ & & \ddots & \\ 0 & & & s \end{pmatrix}$

$Av = sv, \ \forall v \in \mathbb{R}^n$

$$Av = sv \quad , \forall\, v \in \mathbb{R}^n$$

s = egenvärd , v are eigenvectors.

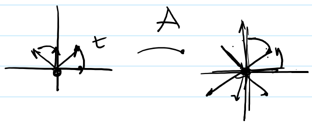
2) $A_{3\times3} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ Diagonal matrix

$$\boxed{Av \overset{?}{=} \lambda v}$$

$$\begin{cases} Ae_1 = ae_1 \\ Ae_2 = be_2 \\ Ae_3 = ce_3 \end{cases}$$

$t = 45°$

3) $\begin{cases} A_{2\times2} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \\ \quad t \neq \{0, \pi\} \\ Av \neq \lambda v \quad \text{for all } v \in \mathbb{R}, \; v \neq 0 \end{cases}$



4) $\underset{2\times2}{A} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$Av = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6-2 \\ 2+0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

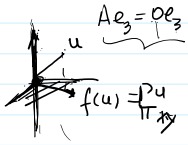
$\boxed{Av = 2v} \qquad \lambda = 2 \;, v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Au = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3-2 \\ 1+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\boxed{Au = 1u} \qquad \boxed{\lambda = 1} \quad u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$Ae_3 = 0e_3$



prop8.6
$A_{n\times n}$ $\quad \boxed{Av = \lambda v}$, $v \neq 0$ $\quad \lambda \in \mathbb{R}$

Then (cv) , $c \in \mathbb{R}$ is also a eigenvector associated to the same λ.

$$A(cv) = c(Av) = c(\lambda v) = \lambda(cv)$$

Satsen. $\boxed{Av = \lambda v}$ ⊛

then v is also eigenvector for A^m, $m \in \mathbb{Z}$.
associated to λ^m

$\to Av = \lambda v$

$$A^2 = A.A \Rightarrow A^2 v = A(\overbrace{Av}^{*}) = A(\lambda v) = \lambda(Av)$$
$$= \lambda(\lambda v) = \lambda^2 v$$

Induktion HYP: $A^{n-1} v = \lambda^{n-1} v$

$$A^n v = A(A^{n-1} v) = A(\lambda^{n-1} v) =$$
$$\lambda^{n-1}(Av) = \lambda^{n-1} \lambda v$$
$$= \lambda^n v$$

$$= \lambda^n v$$

$$Av = \lambda v \qquad \boxed{A^n v = \lambda^n v}$$

ii) $1/\lambda$, v are going the the eigenvalue and eigvektor for A^{-1}

$$I = A \cdot A^{-1} = A^{-1} \cdot A$$

$$v = Iv = (A^{-1} \cdot A)v = A^{-1}(Av) = A^{-1}(\lambda v) = \lambda A^{-1} v$$

$$\boxed{A^{-1} v = \frac{1}{\lambda} v}$$

§ 8.2 Beräkning av Egenvärden och Egenvektorer

$A_{n \times n}$? How to compute λ , v $Av = \lambda v$? $\boxed{v \neq 0}$

$$\boxed{Av = \lambda v} \iff Av - \lambda v = 0 \iff Av - \lambda I_n v = 0 \iff \boxed{(A - \lambda I) v = 0}$$

$A^{-1}Av = A^{-1}\lambda v$

$\boxed{v = \lambda A^{-1} v}$

This system has to have infinite solutions

$\boxed{v \neq 0 \text{ is not allowed}}$

$v \neq 0 \iff \boxed{\det(A - \lambda I) = 0}$

Karacteristika ekvation

λ are the values that satisfy this $\boxed{\det(A - \lambda I) = 0}$

v are going to be solutions of the Homogeneous system for each choice of λ.

$$\underset{2\times 2}{A} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \qquad \begin{gathered} Av = \lambda v \\ (Av - \lambda v) = 0 \\ \boxed{(A - \lambda I)v = 0} \end{gathered}$$

1) $A - \lambda I = \begin{pmatrix} 3-\lambda & -2 \\ 1 & 0-\lambda \end{pmatrix}$

2) $\det(A - \lambda I) = 0 \Rightarrow (3-\lambda)(0-\lambda) - (-2) = 0 \Rightarrow$

$$\Rightarrow -3\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda_{1,2} = \frac{3 \pm \sqrt{9 - 4.2.1}}{2} = \begin{cases} \frac{3+1}{2} = \frac{4}{2} = 2 \\ \frac{3-1}{2} = \frac{2}{2} = 1 \end{cases}$$

$\boxed{\lambda_1 = 2}$, $\boxed{\lambda_2 = 1}$

3) Compute the eigenvector associated to $\lambda_1 = 2$ $\boxed{Av = 2v}$

$$\lambda = 2 \qquad \begin{pmatrix} 3-2 & -2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \quad 1x - 2y = 0 \Rightarrow \boxed{x = 2y}$$

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ y \end{pmatrix} = \cancel{t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\lambda_1 = 2 \longrightarrow v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\lambda_2 = 1$

$$\begin{pmatrix} 3-1 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad x - y = 0 \quad \boxed{x = y}$$

$$v_2 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = \cancel{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$t \in \mathbb{R}$

$$A_E = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \qquad \lambda_1 = 2 \Rightarrow v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \lambda_2 = 1 \Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$v_1 \cdot v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 + 1 = 3$$

v_1 not orthogonal to v_2

but $\{v_1, v_2\}$ linear indep.

Oberoende

$$G = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$E = \{ \bar{e}_1, \bar{e}_2 \}$$

$$\boxed{A_E = G\, A_G\, G^{-1}}$$

$$\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} A_G\, G^{-1} \qquad A_G = ?$$

$$G^{-1} = \frac{1}{\det G} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\boxed{A_G = G^{-1} A_E\, G}$$

$$\det G = 2 - 1 = 1$$

$$A_G = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix} = \left(2\begin{pmatrix} 2 \\ 1 \end{pmatrix} \;\; 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{A_G = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}$$

Ex2 $\quad \underset{2\times 2}{A} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ $\quad$ A symmetric

$$\boxed{Av = \lambda v} \quad \begin{cases} v \neq 0 \\ \lambda \in \mathbb{R} \end{cases}$$

$$Av = \lambda v \Leftrightarrow Av - \lambda v = 0 \Leftrightarrow \boxed{(A - \lambda I)v = 0}$$

$$\boxed{v \neq 0} \Leftrightarrow \boxed{\det(A - \lambda I) = 0}$$

$\boxed{v \neq 0} \Leftrightarrow \boxed{\det(A - \lambda I) = 0}$

1) $\det \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} = 0 \qquad (3-\lambda)(3-\lambda) - 1 = 0$

$9 - 3\lambda - 3\lambda + \lambda^2 - 1 = 0$

$\lambda^2 - 6\lambda + 8 = 0 \qquad \lambda_{1,2} \begin{cases} \boxed{\lambda_1 = 4} \\ \boxed{\lambda_2 = 2} \end{cases}$

2) Different eigenvalues →

$\lambda_1 = 4 \qquad \begin{pmatrix} 3-4 & 1 \\ 1 & 3-4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow x = y \Longrightarrow v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = \overset{t}{\cancel{y}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 2 \qquad \begin{pmatrix} 3-2 & 1 \\ 1 & 3-2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Longrightarrow$

$\begin{cases} x + y = 0 \end{cases} \Longrightarrow \boxed{x = -y}$

$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = \overset{t}{\cancel{y}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$G = \left\{ \underbrace{\overbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}^{v_1}}_{\lambda = 4}, \underbrace{\overbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}^{v_2}}_{\lambda = 2} \right\} \qquad v_1 \cdot v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} =$

$-1 + 1 = 0$

$v_1 \perp v_2$

$A_E = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \qquad G = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$

$\|v_1\| = \sqrt{1+1} = \sqrt{2} \qquad P = \left\{ \underbrace{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}}_{\frac{v_1}{\|v_1\|}} \underbrace{\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}}_{\frac{v_2}{\|v_2\|}} \right\}$

$\frac{v}{\|v_1\|}$

P = ON matrix $\qquad \boxed{P^{-1} = P^t}$

$$\boxed{A_E = P A_G P^{-1}}$$

$A_E = P A_G P^t \qquad P$ = eigenvector base (ON base)

$\boxed{A_E = P D P^t} \qquad P = \{v_1, v_2 \ldots v_n\}$

$A \cdot ((A(v_1) \; (A(v_2)) \; (A(v_n)) = ((\lambda_1 v_1) \; (\lambda_2 v_2) \ldots (\lambda_n v_n)) -$

$$\lambda_1 v_1 = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & & & \lambda_n \\ \vdots & \vdots \end{pmatrix}$$

$\alpha_1 = \lambda_1 \qquad \alpha_2 = \alpha_3 = \dots = \alpha_n = 0$

$$A_E = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \qquad P = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\boxed{A_E = P \underbrace{D}_{A_G} P^t} \qquad P^{-1} = P^t$$

$$P^{-1} A_E = \cancel{P^{-1}P} D P^t$$

$$P^{-1} A_E P = \underbrace{D}_{A_G}$$

$$\boxed{P^t A_E P = D}$$

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§ 8.3 Spektralsatser

Spectrum $S = \{\lambda_1, \lambda_2, \dots \lambda_n\}$

S.8.14 $Av = \lambda v \qquad \lambda_i \neq \lambda_j \Rightarrow \{v_i, v_j\}$ linear independent

8.15. $\boxed{A_{n\times n}}$ $\boxed{\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \dots \neq \lambda_n} \Rightarrow$

$\{v_1, v_2, \dots, v_n\}$ Linear independent set.
$n = \dim \mathbb{R}^n$

$\{v_1, \dots v_n\}$ spanns $\mathbb{R}^n$ &
therefore is a base of $V = \mathbb{R}^n$

8.16 $A_{n\times n}$ symmetrik $\Rightarrow \lambda_i \in \mathbb{R}$ for $i = 1 \dots n$
all eigenvalues are real numbers

8.17 $A_{n\times n}$ symmetrik (*) $\quad Au = \lambda u \quad Av = \mu v \quad \lambda \neq \mu$

$\boxed{u \perp v}$

Bevis $(\lambda u) \cdot v = (\lambda u)^t v = (Au)^t v = (u^t A^t) v =$
$= u^t (A^t v) \overset{(*)}{=} u^t (Av) = u^t (\mu v) =$
$= \mu u^t v = \mu u \cdot v$

$\lambda u \cdot v - \mu u \cdot v = 0$

$(\lambda - \mu)\, u \cdot v = 0$

$\underbrace{\lambda - \mu}_{\lambda-\mu \neq 0} \quad \underbrace{u \cdot v}_{} \longrightarrow u \cdot v = 0 \Longrightarrow \underline{u \perp v}$ $\square$