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SSY281 - MODEL PREDICTIVE CONTROL

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Lecture 5: Setpoints, disturbances and observers

Goals for today:

- To understand how the state model is used for open loop prediction
- To formulate the DMC scheme with a constant output disturbance
- To interpret the DMC scheme in terms of state observers
- To formulate an MPC controller including setpoints and steady state targets
- To state conditions for offset-free control

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Learning objectives:

- Correctly state, in mathematical form, MPC formulations based on descriptions of control problems expressed in application terms
- Describe and construct MPC controllers based on a linear model, quadratic costs and linear constraints
- Describe basic properties of MPC controllers and analyse algorithmic details on very simple examples

Repeat what was used for the batch approach

Let k denote current time and denote by $\hat{x}(k+i|k)$ the estimate of the state $x(k+i)$, *given information available at time* k. By iterating the system equations we get

$$\begin{aligned}
\hat{x}(k+1|k) &= A\hat{x}(k|k) + Bu(k) \\
\hat{x}(k+2|k) &= A\hat{x}(k+1|k) + Bu(k+1) = A^2\hat{x}(k|k) + ABu(k) + Bu(k+1) \\
&\vdots \\
\hat{x}(k+N|k) &= A^N\hat{x}(k|k) + A^{N-1}Bu(k) + A^{N-2}Bu(k+1) + \ldots + Bu(k+N-1).
\end{aligned}$$

State predictions

The state model

$$x(k+1) = Ax(k) + Bu(k)$$

gives the state predictions

$$\begin{bmatrix} \hat{x}(k+1|k) \\ \hat{x}(k+2|k) \\ \vdots \\ \hat{x}(k+N|k) \end{bmatrix} = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} \hat{x}(k|k) + \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix} \begin{bmatrix} u(k|k) \\ u(k+1|k) \\ \vdots \\ u(k+N-1|k) \end{bmatrix}. \tag{42}$$

Output predictions

The state model

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

gives the output predictions

$$\begin{bmatrix} \hat{y}(k+1|k) \\ \hat{y}(k+2|k) \\ \vdots \\ \hat{y}(k+N|k) \end{bmatrix} = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^N \end{bmatrix} \hat{x}(k|k) + \begin{bmatrix} CB & 0 & \cdots & 0 \\ CAB & CB & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{N-1}B & CA^{N-2}B & \cdots & CB \end{bmatrix} \begin{bmatrix} u(k|k) \\ u(k+1|k) \\ \vdots \\ u(k+N-1|k) \end{bmatrix}. \tag{43}$$

Output predictions using control moves

$$
\begin{bmatrix} \hat{y}(k+1|k) \\ \hat{y}(k+2|k) \\ \vdots \\ \hat{y}(k+N|k) \end{bmatrix} = \underbrace{\begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^N \end{bmatrix} \hat{x}(k|k) + \begin{bmatrix} CB & 0 & \cdots & 0 \\ CAB & CB & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{N-1}B & CA^{N-2}B & \cdots & CB \end{bmatrix} \begin{bmatrix} I \\ \vdots \\ \vdots \\ I \end{bmatrix} u(k-1)}_{\text{free response}}
$$

$$
+ \begin{bmatrix} CB & 0 & \cdots & 0 \\ CAB + CB & CB & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^{N-1} CA^i B & \sum_{i=0}^{N-2} CA^i B & \cdots & CB \end{bmatrix} \begin{bmatrix} \Delta u(k|k) \\ \Delta u(k+1|k) \\ \vdots \\ \Delta u(k+N-1|k) \end{bmatrix} \tag{44}
$$

or

$$
\boldsymbol{y}(k) = \boldsymbol{y_f}(k) + \Theta \boldsymbol{\Delta u}(k). \tag{45}
$$

Example: The DMC scheme

Assume that a plant is open loop stable, and that we choose the simplest observer available, namely a pure simulation of a model of the plant. The observer is thus described by the equation[1]

$$\hat{x}(k|k) = A\hat{x}(k-1|k-1) + Bu(k-1),$$

where we assume perfect knowledge of the system matrices A and B. Since A is stable,

$$\hat{x}(k|k) - x(k) \to 0,\ k \to \infty,$$

i.e. the state will asymptotically be estimated perfectly. To stress the fact that predicted outputs are based on a model of the plant, equation (45) is rewritten as

$$\boldsymbol{y}_{\mathcal{M}}(k) = \boldsymbol{y_f}(k) + \Theta\boldsymbol{\Delta u}(k), \tag{46}$$

where superscript $\mathcal{M}$ stands for ‘model’.

[1] The notation $\hat{x}(k|k)$ may seem a bit odd here, since the estimate is not using any information at time k, but since this is a consequence of the deliberate choice of not using the measured output, we stick to the general notation anyway.

Consequences from using a simple observer

1. Note that with the estimator used, the free response y_f depends only on previous control inputs. If there are no (active) constraints, the control action is computed from y_f and possibly a reference trajectory. Hence, the computed control signal does not depend on previous outputs, i.e. *there is no feedback*!

Consequences from using a simple observer

1. Note that with the estimator used, the free response $\boldsymbol{y_f}$ depends only on previous control inputs. If there are no (active) constraints, the control action is computed from $\boldsymbol{y_f}$ and possibly a reference trajectory. Hence, the computed control signal does not depend on previous outputs, i.e. *there is no feedback*!
2. Because of the absence of feedback, we get problems with disturbances. Assume that there is a constant load disturbance d, so that the vector of plant outputs $\boldsymbol{y}_{\mathcal{P}}$ is given by

$$\boldsymbol{y}_{\mathcal{P}}(k) = \boldsymbol{y_f}(k) + \Theta \boldsymbol{\Delta u}(k) + d \cdot \mathbf{1}, \tag{47}$$

where $\mathbf{1}$ is a vector of 1's. In steady-state, the control signal will be determined so that the model output equals the reference, i.e.

$$\boldsymbol{y}_{\mathcal{P}}(k) = \boldsymbol{y}_{\mathcal{M}}(k) + d \cdot \mathbf{1} = \boldsymbol{r}(k) + d \cdot \mathbf{1},$$

where we have used the notation $\boldsymbol{r}$ for the vector of reference values introduced in Example 1.2. The conclusion is that there will be a steady-state error.

Removing steady-state error by the DMC scheme

Start by changing the model to

$$y(k) = Cx(k) + \hat{d},$$

where $\hat{d}$ is the (unknown) constant disturbance. Now, estimate this disturbance simply by computing

$$\hat{d}(k|k) = y(k) - C\hat{x}(k|k) = y(k) - C\left(A\hat{x}(k-1|k-1) + Bu(k-1)\right)$$
$$\hat{d}(k+i|k) = \hat{d}(k|k), \quad i = 1, 2, \ldots$$

Removing steady-state error by the DMC scheme

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This will instead of (46) result in

$$\boldsymbol{y}_{\mathcal{M}}(k) = \boldsymbol{y}_{\boldsymbol{f}}(k) + \Theta\boldsymbol{\Delta u}(k) + \hat{d}(k|k) \cdot \mathbf{1}.$$

The consequence is that in steady state (and if $\hat{d} \to d$) we will now have $\boldsymbol{y}_{\mathcal{P}} = \boldsymbol{y}_{\mathcal{M}} = \boldsymbol{r}$. Note that the modification done has now introduced feedback into the controller!

State observer

Given a system model

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k), \end{aligned}$$

with observer update

$$\hat{x}(k+1|k) = A\hat{x}(k|k-1) + Bu(k) + L(y(k) - C\hat{x}(k|k-1))$$

and error dynamics

$$\begin{aligned} \tilde{x}(k) &= \hat{x}(k) - x(k) \\ \tilde{x}(k+1) &= (A - LC)\tilde{x}(k). \end{aligned}$$

Example: The DMC scheme, cont'd

The assumption that the plant is affected by a constant load disturbance d can be expressed as an extension of the original model,

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) \\
d(k+1) &= d(k) \\
y(k) &= Cx(k) + d(k).
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Example: The DMC scheme, cont'd

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y(k) &= Cx(k) + d(k).
\end{aligned}
$$

By introducing the augmented state

$$\xi(k) = \begin{bmatrix} x(k) \\ d(k) \end{bmatrix}$$

we get the new model

$$
\begin{aligned}
\xi(k+1) &= \mathcal{A}\xi(k) + \mathcal{B}u(k) \\
y(k) &= \mathcal{C}\xi(k)
\end{aligned}
$$

with

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \quad \mathcal{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \quad \mathcal{C} = \begin{bmatrix} C & I \end{bmatrix}.$$

By defining the observer gain as $L = \begin{bmatrix} L_x \\ L_d \end{bmatrix}$, a standard observer for the extended model is given by

$$\begin{aligned}
\hat{\xi}(k+1) &= \mathcal{A}\hat{\xi}(k) + \mathcal{B}u(k) + L(y(k) - \mathcal{C}\hat{\xi}(k)) \\
&= \left(\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} L_x \\ L_d \end{bmatrix} \begin{bmatrix} C & I \end{bmatrix} \right) \hat{\xi}(k) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} L_x \\ L_d \end{bmatrix} y(k).
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The observer dynamics is determined by the first big matrix in the last expression. With the simple choice $L_x = 0$ and $L_d = I$, we get the observer error dynamics given by

$$\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} L_x \\ L_d \end{bmatrix} \begin{bmatrix} C & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}.$$

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The eigenvalues of this matrix are given by the matrix A, which was assumed to be stable, and the rest of the eigenvalues are located in the origin, a choice that is usually referred to as *deadbeat* dynamics.

MPC overall structure

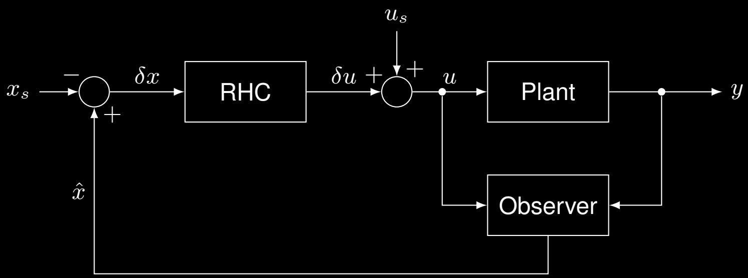

Figure 9: MPC overall structure with a receding horizon controller (RHC) working on deviation variables.

Steady-state target problem ($p = m$)

If the system is square ($p = m$):

1. Define desired setpoints for the outputs, y_{sp}.

Steady-state target problem ($p = m$)

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1. Define desired setpoints for the outputs, y_{sp}.
2. Solve the following system of linear equations to find the steady-state targets

$$\begin{bmatrix} I - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} B_d \hat{d} \\ y_{sp} - C_d \hat{d} \end{bmatrix}. \tag{48}$$

Main computational tasks to be carried out by the MPC

1. Update the state estimate (including load disturbance).

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2. Compute an updated steady state target, e.g. by solving equation (48).
3. Compute the next control action by solving a constrained optimisation problem.

MPC block diagram

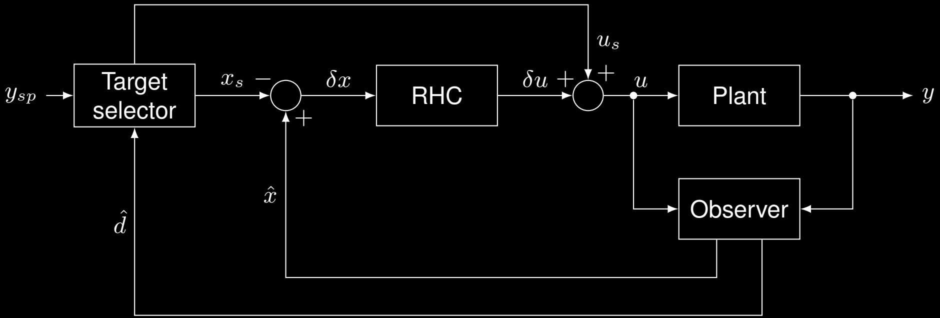

Figure 10: MPC block diagram with a receding horizon controller (RHC) working on deviation variables.

Steady-state target problem ($p > m$)

If there are more outputs than inputs ($p > m$):

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1. Define desired setpoints for the outputs, y_{sp}.
2. Solve the following optimisation problem to find the best steady-state targets:

$$\min_{x_s, u_s} \left(|Cx_s - y_{sp}|_Q^2 \right), \quad Q \succeq 0$$

subject to

$$\begin{bmatrix} I - A & -B \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = 0$$

$$Eu_s \leq e$$

$$FCx_s \leq f.$$

Steady-state target problem ($p_z < m$)

If there are more inputs than controlled outputs ($p_z < m$):

1. Define setpoints for controlled outputs, z_{sp}, and desired values for the control input, u_{sp}, and non-controlled outputs y_{sp}.

Steady-state target problem ($p_z < m$)

If there are more inputs than controlled outputs ($p_z < m$):

1. Define setpoints for controlled outputs, z_{sp}, and desired values for the control input, u_{sp}, and non-controlled outputs y_{sp}.
2. Solve the following optimisation problem to find feasible steady-state targets:

$$\min_{x_s, u_s} \left(|u_s - u_{sp}|^2_{R_s} + |C_y x_s - y_{sp}|^2_{Q_s} \right), \quad R_s \succ 0$$

subject to

$$\begin{bmatrix} I - A & -B \\ C_z & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ z_{sp} \end{bmatrix}$$

$$E u_s \leq e$$

$$F C_z x_s \leq f.$$

Steady-state target problem with disturbance

Optimisation problem to find feasible steady-state target:

$$\min_{x_s, u_s} \left(|u_s - u_{sp}|^2_{R_s} + |Cx_s + C_d\hat{d} - y_{sp}|^2_{Q_s} \right), \quad R_s \succ 0$$

subject to

$$\begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} B_d\hat{d} \\ z_{sp} - HC_d\hat{d} \end{bmatrix}$$

$$Eu_s \leq e$$

$$FHCx_s \leq f - FHC_d\hat{d}.$$

Off-set free control

Proposition (Off-set free control)

Assume that the steady-state target problem is feasible and that an MPC with the following augmented model is used:

$$\begin{aligned} \begin{bmatrix} x \\ d \end{bmatrix}^+ &= \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix}. \end{aligned} \tag{49}$$

Further assume that $C_z = HC$, $n_d = p$ *and that* B_d, C_d *are chosen such that*

$$\operatorname{rank} \begin{bmatrix} I - A & -B_d \\ C & C_d \end{bmatrix} = n + p. \tag{50}$$

Assume that the closed-loop converges to a steady-state with constraints inactive. Then there is zero off-set in the controlled outputs, i.e.

$$z_s = z_{sp}.$$

References

[1] J.B. Rawlings, D.Q. Mayne, and M.M. Diehl. *Model Predictive Control: Theory, Computation, and Design, 2nd edition*. Nob Hill Publishing 2017.
Available online at `https://sites.engineering.ucsb.edu/~jbraw/mpc`

[2] G. Goodwin, M.M. Seron, and J.A. De Don·. *Constrained Control and Estimation*. Springer 2004.
Available online via Chalmers Library.

[3] F Borrelli, A. Bemporad, and M. Morari. *Predictive Control for Linear and Hybrid Systems*.
Available online at http://www.mpc.berkeley.edu/mpc-course-material

[4] J. Maciejowski. *Predictive Control with Constraints*. Prentice Hall 2002.

[5] S. Boyd and L. Vandenberghe. *Convex optimisation*. Cambridge University Press 2004.

[6] Diehl, M. *Real-Time Optimization for Large Scale Nonlinear Processes.* PhD thesis, University of Heidelberg, 2001.

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