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UNIVERSITY OF TECHNOLOGY

# SSY281 - MODEL PREDICTIVE CONTROL

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# Lecture 6: The Kalman filter and moving horizon estimation

**Goals for today:**

- To refresh the Kalman filter
- To formulate a state estimator based on least-squares (LS)
- To formulate a moving horizon estimator based on LS
- To formulate a moving horizon estimator with constraints

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- To refresh the Kalman filter
- To formulate a state estimator based on least-squares (LS)
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Learning objectives:

- Understand and explain the basic principles of model predictive control, its pros and cons, and the challenges met in implementation and applications
- Correctly state, in mathematical form, MPC formulations based on descriptions of control problems expressed in application terms
- Describe and construct MPC controllers based on a linear model, quadratic costs and linear constraints

# The Kalman filter

System model:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + w(k), \quad x(0) \sim \mathcal{N}(x_0, P_0), \quad w \sim \mathcal{N}(0, Q) \\ y(k) &= Cx(k) + v(k), \quad v \sim \mathcal{N}(0, R). \end{aligned}$$

State estimator:

Correction: $\hat{x}(k|k) = \hat{x}(k|k-1) + L(k)[y(k) - C\hat{x}(k|k-1)], \quad \hat{x}(0|0) = x_0$

Prediction: $\hat{x}(k+1|k) = A\hat{x}(k|k) + Bu(k).$

Kalman filter gain:

$$L(k) = P(k)C^\top [CP(k)C^\top + R]^{-1}.$$

State estimation error covariance update (note that $P(k) \equiv P(k|k-1)$):

Estimation error: $P(k|k) = P(k) - P(k)C^\top [CP(k)C^\top + R]^{-1} CP(k), \; P(0|0) = P_0$

Prediction error: $P(k+1) = AP(k|k)A^\top + Q.$

# Stationary Kalman filter

Consider the linear system

$$
\begin{aligned}
x^+ &= Ax + Bu + w, \quad w \sim \mathcal{N}(0, Q) \\
y &= Cx + v, \quad v \sim \mathcal{N}(0, R).
\end{aligned}
$$

If the pair $(C, A)$ is observable and $Q, R \succ 0$, then the Kalman filter gain $L(k)$ and the prediction error covariance $P(k)$ converge to the solution of the (filtering) algebraic Riccati equation

$$
\begin{aligned}
L &= PC^\top[CPC^\top + R]^{-1} \\
P &= APA^\top - APC^\top[CPC^\top + R]^{-1}CPA^\top + Q.
\end{aligned}
$$

# Duality of LQR and Kalman filter

LQ regulator:

$$
\begin{aligned}
u(k) &= K(k)x(k), \quad k = 0, \ldots, N-1 \\
K(k) &= -(R + B^\top P(k+1)B)^{-1} B^\top P(k+1)A.
\end{aligned}
$$

Riccati equation:

$$
P(k-1) = Q + A^\top P(k)A - A^\top P(k)B[R + B^\top P(k)B]^{-1} B^\top P(k)A, \quad P(N) = P_f.
$$

Kalman filter:

$$
\begin{aligned}
\hat{x}(k+1|k) &= A\hat{x}(k|k-1) + Bu(k) + L(k)[y(k) - C\hat{x}(k|k-1)] \\
L(k) &= P(k)C^\top [CP(k)C^\top + R]^{-1}.
\end{aligned}
$$

Riccati equation:

$$
P(k+1) = AP(k)A^\top + Q - AP(k)C^\top [CP(k)C^\top + R]^{-1} CP(k)A^\top, \quad P(0) = P_0.
$$

# Least squares estimation

System model:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k). \end{aligned}$$

Optimisation problem:

$$\min_{x(0:T)} V_T(x(0:T))$$

where the minimisation is with respect to the sequence of state estimates

$$x(0:T) = \{x(0), x(1), \ldots, x(T)\}.$$

The objective function $V_T$ is given by

$$\begin{aligned} V_T(x(0:T)) = (x(0) - x_0)^\top P_0^{-1}(x(0) - x_0) + \sum_{i=0}^{T}(y(i) - Cx(i))^\top R^{-1}(y(i) - Cx(i)) \\ + \sum_{i=0}^{T-1}(x(i+1) - Ax(i) - Bu(i))^\top Q^{-1}(x(i+1) - Ax(i) - Bu(i)). \end{aligned} \tag{51}$$

# Dynamic programming solution

$$\min_{x(0:T)} V_T(x(0:T)) =$$

$$\min_{x(1:T)} \left\{ \min_{x(0)} \left( \overbrace{(x(0)-x_0)^\top P_0^{-1}(x(0)-x_0) + (y(0)-Cx(0))^\top R^{-1}(y(0)-Cx(0))}^{V_0(x(0))} \right.\right.$$

$$\left.\left. + \underbrace{(x(1)-Ax(0)-Bu(0))^\top Q^{-1}(x(1)-Ax(0)-Bu(0))}_{V_0(x(0),x(1))-V_0(x(0))} \right) + V(x(1:T)) \right\}, \quad (52)$$

where $V(x(1:T))$ is the objective function $V_T(x(0:T))$ without the terms depending on $x(0)$. *For notational convenience, we will in the derivation assume there is no control signal, i.e.* $u(\cdot) = 0$*; it is straightforward to include* $u$ *in the final expressions.*

# Matrix inversion lemma

## Lemma (Matrix inversion lemma)

*Assuming the invertability of the matrices involved, the following identity holds:*

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B[DA^{-1}B + C^{-1}]^{-1}DA^{-1}.$$

# Step 0: measurement update

Consider the first two terms in (52), i.e. the terms corresponding to the a priori guess $x_0$ and the measurement $y(0)$,

$$\begin{aligned} V_0(x(0)) &= (x(0) - x_0)^\top P_0^{-1} (x(0) - x_0) + (y(0) - Cx(0))^\top R^{-1} (y(0) - Cx(0)) \\ &= x^\top(0)[P_0^{-1} + C^\top R^{-1} C]x(0) - 2x^\top(0)[P_0^{-1}x_0 + C^\top R^{-1} y(0)] + x_0^\top P_0^{-1} x_0 + y^\top(0) R^{-1} y(0). \end{aligned} \tag{53}$$

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Define (the subscript $m$ stands for 'measurement update')

$$P_m(0) = [P_0^{-1} + C^\top R^{-1} C]^{-1} \tag{54}$$

and using the matrix inversion formula with $A \to P_0^{-1}$, $B \to C^\top$, $C \to R^{-1}$, and $D \to C$, we get the relation

$$P_m(0) = P_0 - P_0 C^\top [C P_0 C^\top + R]^{-1} C P_0. \tag{55}$$

Now, by completing the squares we can rewrite equation (53) as

$$
\begin{aligned}
V_0(x(0)) &= x^\top(0)P_m^{-1}(0)x(0) - 2x^\top(0)\left[P_0^{-1}x_0 + C^\top R^{-1}y(0)\right] + x_0^\top P_0^{-1}x_0 + y^\top(0)R^{-1}y(0) \\
&= (x(0)-a)^\top P_m^{-1}(0)\,(x(0)-a) - a^\top P_m^{-1}(0)a + x_0^\top P_0^{-1}x_0 + y^\top(0)R^{-1}y(0),
\end{aligned}
$$

where the vector $a$ is given by

$$
a = P_m(0)[P_0^{-1}x_0 + C^\top R^{-1}y(0)]
$$

Now, by completing the squares we can rewrite equation (53) as

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where the vector $a$ is given by

$$\begin{aligned} a &= P_m(0)[P_0^{-1}x_0 + C^\top R^{-1}y(0)] \\ &= P_m(0)[(P_m^{-1}(0) - C^\top R^{-1}C)x_0 + C^\top R^{-1}y(0)] \end{aligned}$$

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where the vector $a$ is given by

$$\begin{aligned}a &= P_m(0)[P_0^{-1}x_0 + C^\top R^{-1}y(0)] \\ &= P_m(0)[(P_m^{-1}(0) - C^\top R^{-1}C)x_0 + C^\top R^{-1}y(0)] \\ &= x_0 + P_m(0)C^\top R^{-1}(y(0) - Cx_0)\end{aligned}$$

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and equation (55) has been used in the fourth line.

Now, by completing the squares we can rewrite equation (53) as

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&= (x(0)-a)^\top P_m^{-1}(0)\,(x(0)-a) - a^\top P_m^{-1}(0)a + x_0^\top P_0^{-1}x_0 + y^\top(0)R^{-1}y(0),
\end{aligned}
$$

where the vector $a$ is given by

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\begin{aligned}
a &= P_m(0)[P_0^{-1}x_0 + C^\top R^{-1}y(0)] \\
&= P_m(0)[(P_m^{-1}(0) - C^\top R^{-1}C)x_0 + C^\top R^{-1}y(0)] \\
&= x_0 + P_m(0)C^\top R^{-1}(y(0) - Cx_0) \\
&= x_0 + (P_0 - P_0C^\top[CP_0C^\top + R]^{-1}CP_0)C^\top R^{-1}(y(0) - Cx_0) \\
&= x_0 + P_0C^\top(R^{-1} - [CP_0C^\top + R]^{-1}CP_0C^\top R^{-1})(y(0) - Cx_0)
\end{aligned}
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&= P_m(0)[(P_m^{-1}(0) - C^\top R^{-1}C)x_0 + C^\top R^{-1}y(0)] \\
&= x_0 + P_m(0)C^\top R^{-1}(y(0) - Cx_0) \\
&= x_0 + (P_0 - P_0C^\top[CP_0C^\top + R]^{-1}CP_0)C^\top R^{-1}(y(0) - Cx_0) \\
&= x_0 + P_0C^\top(R^{-1} - [CP_0C^\top + R]^{-1}CP_0C^\top R^{-1})(y(0) - Cx_0) \\
&= x_0 + P_0C^\top[CP_0C^\top + R]^{-1}([CP_0C^\top + R]R^{-1} - CP_0C^\top R^{-1})(y(0) - Cx_0)
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&= x_0 + P_0C^\top(R^{-1} - [CP_0C^\top + R]^{-1}CP_0C^\top R^{-1})(y(0) - Cx_0) \\
&= x_0 + P_0C^\top[CP_0C^\top + R]^{-1}([CP_0C^\top + R]R^{-1} - CP_0C^\top R^{-1})(y(0) - Cx_0) \\
&= x_0 + P_0C^\top[CP_0C^\top + R]^{-1}(y(0) - Cx_0),
\end{aligned}$$

and equation (55) has been used in the fourth line.

Defining

$$L(0) = P_0 C^\top [C P_0 C^\top + R]^{-1}$$
$$\hat{x}(0) = a = x_0 + L(0)(y(0) - C x_0),$$

we can now write

$$V_0(x(0)) = (x(0) - \hat{x}(0))^\top P_m^{-1}(0)(x(0) - \hat{x}(0)) + c,$$

where the constant $c$ can be shown to be given by

$$c = (y(0) - C x_0)^\top \left[C P_0 C^\top + R\right]^{-1} (y(0) - C x_0)\,,$$

i.e., it is independent of $x(0)$. We will therefore simply drop the term $c$ from the objective in the sequel.

# Step 0: time update

We now add the remaining term that depends on $x(0)$, denoting the result $V(x(0), x(1))$:

$$\begin{aligned} V(x(0), x(1)) &= V_0(x(0)) + (x(1) - Ax(0))^\top Q^{-1} (x(1) - Ax(0)) \\ &= (x(0) - \hat{x}(0))^\top P_m^{-1}(0) (x(0) - \hat{x}(0)) + (x(1) - Ax(0))^\top Q^{-1} (x(1) - Ax(0)) \,. \end{aligned}$$

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This equation is analogous to equation (53); the ‘translation’ is

$$x_0 \to \hat{x}(0), \quad P_0^{-1} \to P_m^{-1}(0), \quad y(0) \to x(1), \quad C \to A, \quad R^{-1} \to Q^{-1}.$$

We can therefore repeat the procedure involving completing of squares and leading to

$$V(x(0), x(1)) = (x(0) - b)^\top \left[P_m^{-1}(0) + A^\top Q^{-1} A\right]^{-1} (x(0) - b) + d$$
$$b = \hat{x}(0) + P_m(0) A^\top [A P_m(0) A^\top + Q]^{-1} (x(1) - A\hat{x}(0)),$$

where $d$ is a constant, independent of $x(0)$, and given by

$$d = (x(1) - A\hat{x}(0))^\top \left[A P_m(0) A^\top + Q\right]^{-1} (x(1) - A\hat{x}(0)) .$$

We can therefore repeat the procedure involving completing of squares and leading to

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where $d$ is a constant, independent of $x(0)$, and given by

$$d = (x(1) - A\hat{x}(0))^\top \left[AP_m(0)A^\top + Q\right]^{-1} (x(1) - A\hat{x}(0)) \,.$$

It is now immediately seen that $x(0) = b$ minimizes $V(x(0), x(1))$. Denoting the optimal solution by $x^*(0)$ and the minimum by $V_1^-$, we thus have

$$x^*(0) = \hat{x}(0) + P_m(0)A^\top[AP_m(0)A^\top + Q]^{-1} (x(1) - A\hat{x}(0)) \tag{56}$$
$$V_1^- = \min_{x(0)} V(x(0), x(1)) = d = (x(1) - A\hat{x}(0))^\top P_t^{-1}(1) (x(1) - A\hat{x}(0))$$

where we introduced $P_t(1) = AP_m(0)A^\top + Q$ ($t$ stands for ‘time update’).

# Step 1: measurement update

First include the new measurement $y(1)$ in the objective,

$$\begin{aligned} V_1(x(1)) &= V_1^-(x(1)) + (y(1) - Cx(1))^\top R^{-1} (y(1) - Cx(1)) \\ &= (x(1) - A\hat{x}(0))^\top P_t^{-1}(1) (x(1) - A\hat{x}(0)) + (y(1) - Cx(1))^\top R^{-1} (y(1) - Cx(1)) . \end{aligned}$$

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This expression looks exactly like the one we started with in step 0, equation (53). We can therefore only change the variable names and get analogous expressions for these, imitating what was done in step 0:

$$\begin{aligned} P_m(1) &= P_t(1) - P_t(1)C^\top [CP_t(1)C^\top + R]^{-1} CP_t(1) \\ L(1) &= P_t(1)C^\top [CP_t(1)C^\top + R]^{-1} \\ \hat{x}(1) &= A\hat{x}(0) + L(1)(y(1) - CA\hat{x}(0)). \end{aligned}$$

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First include the new measurement $y(1)$ in the objective,

$$\begin{aligned} V_1(x(1)) &= V_1^-(x(1)) + (y(1) - Cx(1))^\top R^{-1} (y(1) - Cx(1)) \\ &= (x(1) - A\hat{x}(0))^\top P_t^{-1}(1) (x(1) - A\hat{x}(0)) + (y(1) - Cx(1))^\top R^{-1} (y(1) - Cx(1)) . \end{aligned}$$

This expression looks exactly like the one we started with in step 0, equation (53). We can therefore only change the variable names and get analogous expressions for these, imitating what was done in step 0:

$$\begin{aligned} P_m(1) &= P_t(1) - P_t(1)C^\top [CP_t(1)C^\top + R]^{-1} CP_t(1) \\ L(1) &= P_t(1)C^\top [CP_t(1)C^\top + R]^{-1} \\ \hat{x}(1) &= A\hat{x}(0) + L(1)(y(1) - CA\hat{x}(0)). \end{aligned}$$

Similarly, the part of the objective function studied can be written

$$V_1(x(1)) = (x(1) - \hat{x}(1))^\top P_m^{-1}(1)(x(1) - \hat{x}(1)). \tag{57}$$

# Least squares estimator

System model:

$$x(k+1) = Ax(k) + Bu(k)), \quad y(k) = Cx(k).$$

Objective function:

$$V_T(x(0\!:\!T)) = (x(0) - x_0)^\top P_0^{-1}(x(0) - x_0) + \sum_{i=0}^{T}(y(i) - Cx(i))^\top R^{-1}(y(i) - Cx(i))$$
$$+ \sum_{i=0}^{T-1}(x(i+1) - Ax(i) - Bu(i))^\top Q^{-1}(x(i+1) - Ax(i) - Bu(i)).$$

State estimator:

$$\begin{aligned}
P_t(k) &= AP_m(k-1)A^\top + Q, \quad P_t(0) = P_0 \\
L(k) &= P_t(k)C^\top[CP_t(k)C^\top + R]^{-1} \\
P_m(k) &= P_t(k) - P_t(k)C^\top[CP_t(k)C^\top + R]^{-1}CP_t(k) \\
\hat{x}(k) &= A\hat{x}(k-1) + Bu(k-1) + L(k)\left(y(k) - C(A\hat{x}(k-1) + Bu(k-1)\right);\ \hat{x}(-1) = x_0 \\
x^*(k) &= \hat{x}(k) + P_m(k)A^\top[AP_m(k)A^\top + Q]^{-1}\left(x^*(k+1) - A\hat{x}(k)\right);\ x^*(T) = \hat{x}(T).
\end{aligned} \tag{58}$$

# Least squares estimator convergence

## Lemma (Least squares estimator convergence)

*Assume that the LS estimator is applied to the system given by*

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k)) \\ y(k) &= Cx(k). \end{aligned}$$

*If the pair* $(C, A)$ *is observable and* $Q, R \succ 0$*, then the LS state estimate converges to the true system state*

$$x^*(T|T) \rightarrow x(T), \quad T \rightarrow \infty.$$

# Moving horizon estimation

System model:

$$\begin{aligned} x(i+1) &= Ax(i) + Bu(i) \\ y(i) &= Cx(i). \end{aligned}$$

Optimisation problem:

$$\min_{x(k-T:k)} \hat{V}_T(x(k-T:k))$$

where the minimisation is with respect to the sequence of state estimates

$$x(k-T:k) = \{x(k-T), x(k-T+1), \ldots, x(k)\}.$$

The objective function $V_T$ is given by

$$\begin{aligned} \hat{V}_T(x(k-T:k)) = \sum_{i=k-T}^{k-1} &(x(i+1) - Ax(i) - Bu(i))^\top Q^{-1} (x(i+1) - Ax(i) - Bu(i)) \\ + \sum_{i=k-T}^{k} &(y(i) - Cx(i))^\top R^{-1} (y(i) - Cx(i)) \,. \end{aligned} \tag{59}$$

# Constrained moving horizon estimator

Objective function:

$$\hat{V}_T(x(k-T:k)) = \sum_{i=k-T}^{k-1} \omega(i)^\top Q^{-1} \omega(i) + \sum_{i=k-T}^{k} \nu(i)^\top R^{-1} \nu(i).$$

Optimisation problem:

$$\min \hat{V}_T(x(k-T:k))$$

with respect to

$$\{x(k-T:k),\, \omega(k-T:k-1),\, \nu(k-T:k)\}$$

and subject to

$$\omega(i) = x(i+1) - (Ax(i) + Bu(i)), \quad x(0) = x_0$$
$$\nu(i) = y(i) - Cx(i)$$
$$x(i) \in \mathbb{X}, \quad \omega(i) \in \mathbb{W}, \quad \nu(i) \in \mathbb{V}, \quad \text{for all } i \in (k-T, k).$$

# Constrained moving horizon estimator with non-zero prior weighting

Objective function:

$$\hat{V}_T(x(k-T:k)) = (x(k-T) - x_{k-T})^\top P_0^{-1}(x(k-T) - x_{k-T}) + \sum_{i=k-T}^{k-1} \omega(i)^\top Q^{-1}\omega(i) + \sum_{i=k-T}^{k} \nu(i)^\top R^{-1}\nu(i).$$

Optimisation problem:

$$\min \hat{V}_T(x(k-T:k))$$

with respect to

$$\{x(k-T:k),\ \omega(k-T:k-1),\ \nu(k-T:k)\}$$

and subject to

$$\omega(i) = x(i+1) - (Ax(i) + Bu(i))$$
$$\nu(i) = y(i) - Cx(i)$$
$$x(i) \in \mathbb{X}, \quad \omega(i) \in \mathbb{W}, \quad \nu(i) \in \mathbb{V}, \quad \text{for all } i \in (k-T, k).$$

# References

[1] J.B. Rawlings, D.Q. Mayne, and M.M. Diehl. *Model Predictive Control: Theory, Computation, and Design, 2nd edition*. Nob Hill Publishing 2017.
Available online at `https://sites.engineering.ucsb.edu/~jbraw/mpc`

[2] G. Goodwin, M.M. Seron, and J.A. De Don·. *Constrained Control and Estimation*. Springer 2004.
Available online via Chalmers Library.

[3] F Borrelli, A. Bemporad, and M. Morari. *Predictive Control for Linear and Hybrid Systems*.
Available online at http://www.mpc.berkeley.edu/mpc-course-material

[4] J. Maciejowski. *Predictive Control with Constraints*. Prentice Hall 2002.

[5] S. Boyd and L. Vandenberghe. *Convex optimisation*. Cambridge University Press 2004.

[6] Diehl, M. *Real-Time Optimization for Large Scale Nonlinear Processes*. PhD thesis, University of Heidelberg, 2001.

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