CHALMERS
UNIVERSITY OF TECHNOLOGY

SSY281 - MODEL PREDICTIVE CONTROL

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Lecture 8 : Solving QP problems

Goals for today:

- To formulate Newton's method to solve the KKT conditions in simple cases
- To understand the principles of active set and interior point methods for QP:s

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Learning objectives:

- Understand and explain basic properties of the optimisation problem as an ingredient of MPC, in particular concepts like linear, quadratic and convex optimisation, optimality conditions, and feasibility

Newton's method

In its simplest form, the Newton's method is used to find the root of the scalar equation $r(x) = 0$. The idea is to approximate the function by a straight line at the current "guess" x, and to obtain the next guess[2] $x^+ = x + \Delta x$ as the root of the linear approximation

$$r(x + \Delta x) \approx r(x) + r'(x)\Delta x = 0 \quad \Rightarrow \quad \Delta x = -(r'(x))^{-1} r(x). \tag{69}$$

When x and $r(x)$ are vectors, the *Newton step* is a direct generalization of this, i.e.

$$\nabla r(x)^\top \Delta x = -r(x), \tag{70}$$

which is now a system of *linear* equations in the unknowns Δx. The new iterate can be obtained as

$$x^+ = x + \Delta x.$$

[2] The notation x^+ is used here for the next iterate, not a time update.

An illustration of the Newton method

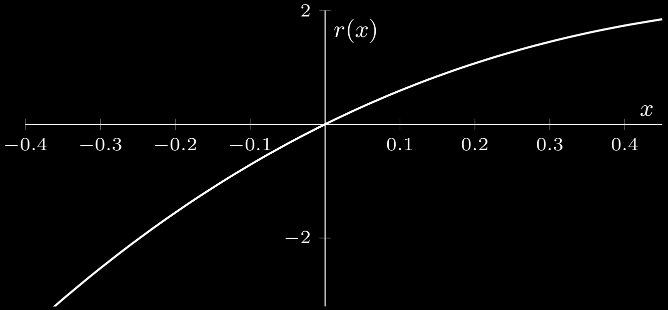
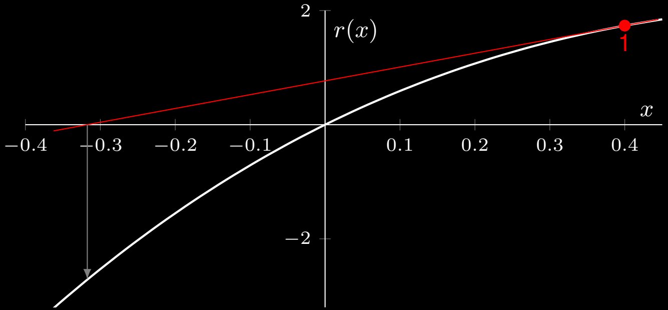


Figure 20: An illustration of the Newton method using a full step size. The method converges in about 4–6 steps, with a roughly quadratic convergence rate.

An illustration of the Newton method

1. $r = 1.7360$



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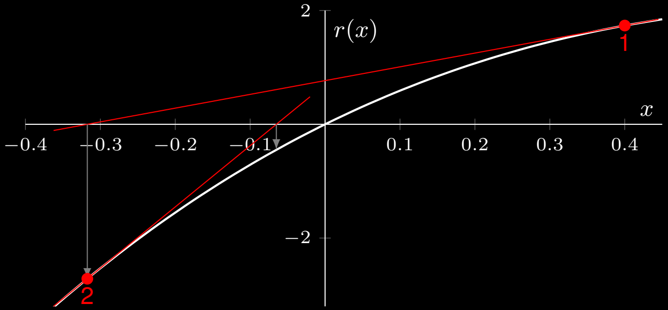
1. $r = 1.7360$
2. $r = -2.7150$



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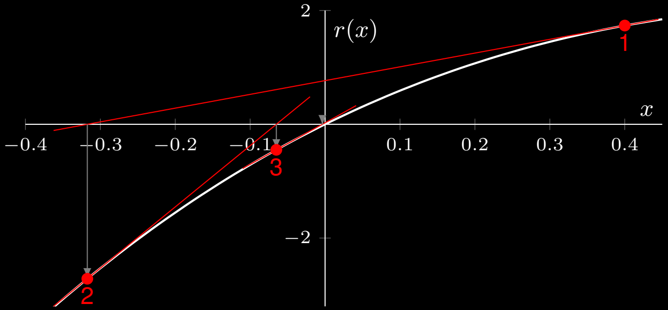
1. $r = 1.7360$
2. $r = -2.7150$
3. $r = -0.4487$



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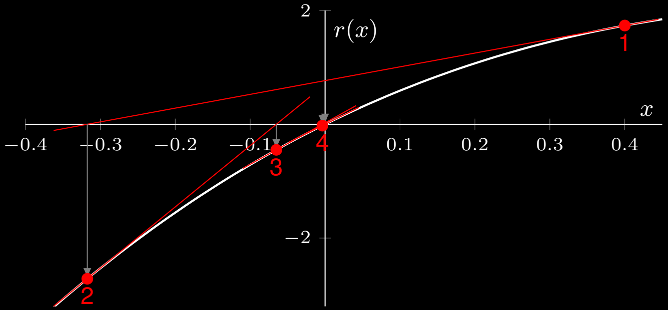
An illustration of the Newton method

1. $r = 1.7360$
2. $r = -2.7150$
3. $r = -0.4487$
4. $r = -0.0234$
5. $r = -0.0001$
6. $r = -8.43 \times 10^{-10}$



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Newton's method with reduced step

Since Newton's method is based on a linear approximation of $r(x)$, it turns out that it is usually wise in practice not to perform the *full Newton step* as given by (70). Instead, while still going in the *Newton direction* as prescribed by (70), the new iterate is obtained by using a *reduced step size*:

$$x^+ = x + t\Delta x, \quad t \in (0, 1]. \tag{71}$$

Failure of Newton method

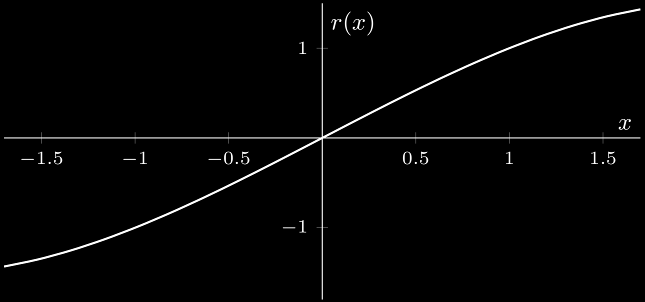
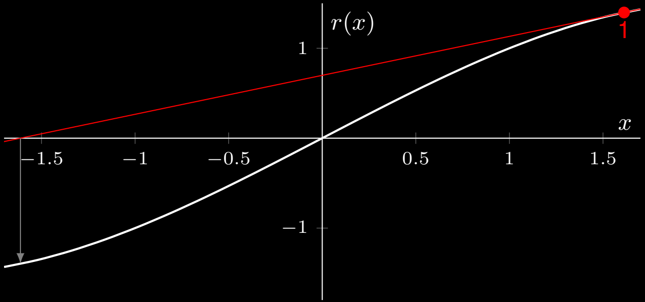


Figure 21: Failure of the Newton method to converge when using a full step size.

Failure of Newton method



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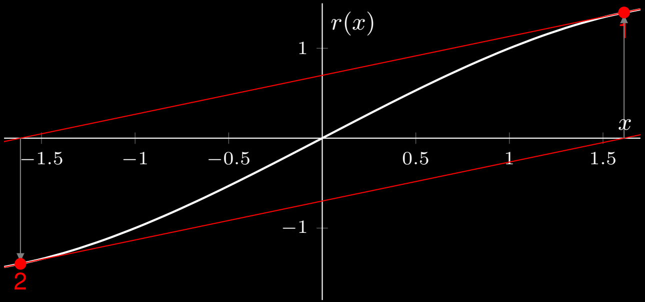

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Newton method with a reduced step-size

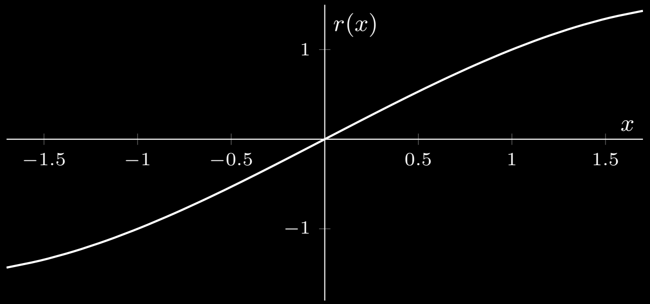
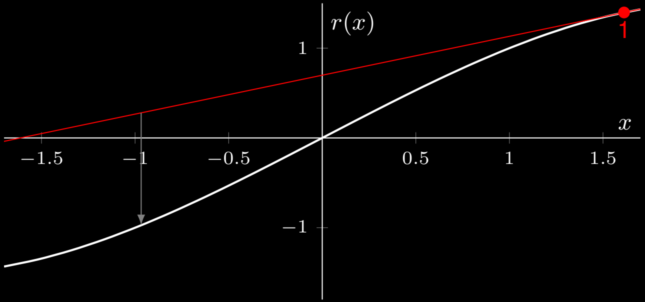


Figure 22: The Newton method using a reduced step size of 0.8. The method converges in about four steps.

Newton method with a reduced step-size



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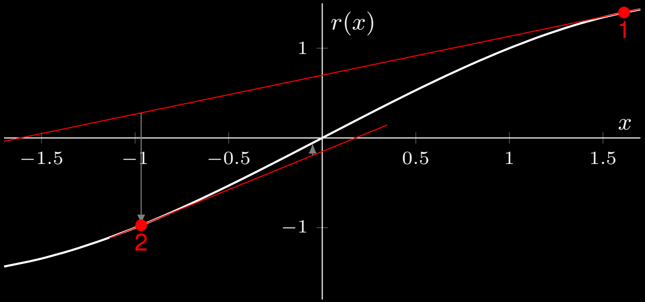

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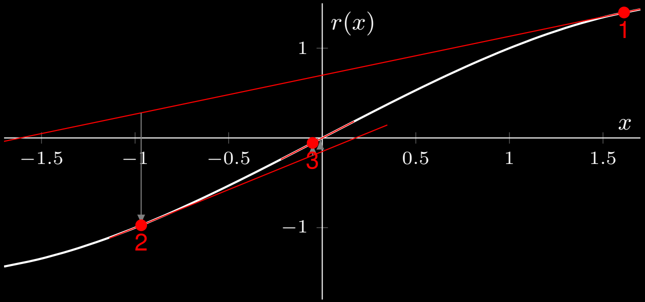

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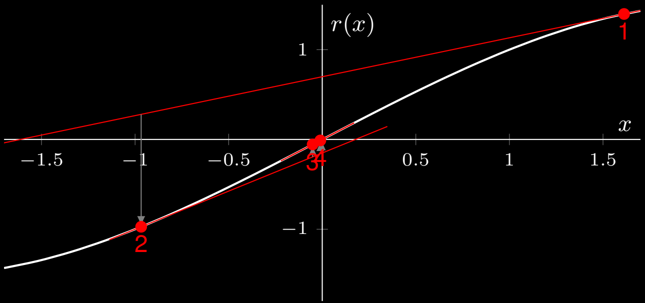

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Failure of Newton method: convergence to a statinary point

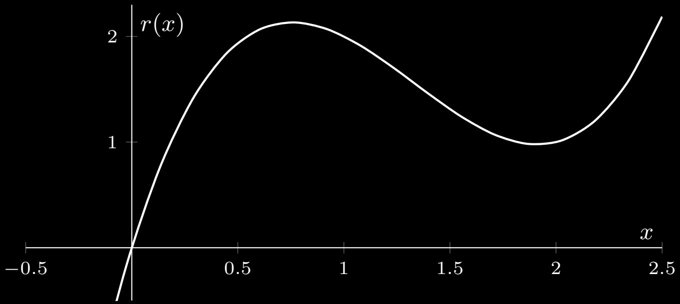

Figure 23: The Newton method converges to a statinary point where derivative is zero. The method loses a search direction and is not able to solve the problem.

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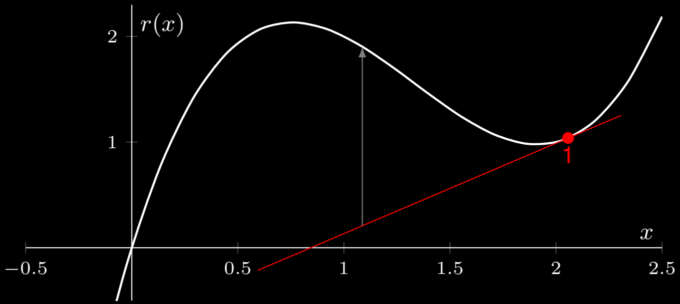

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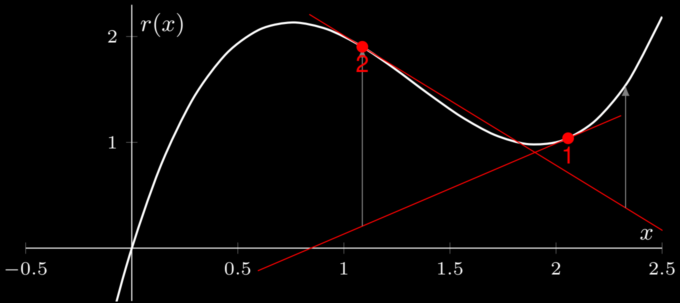

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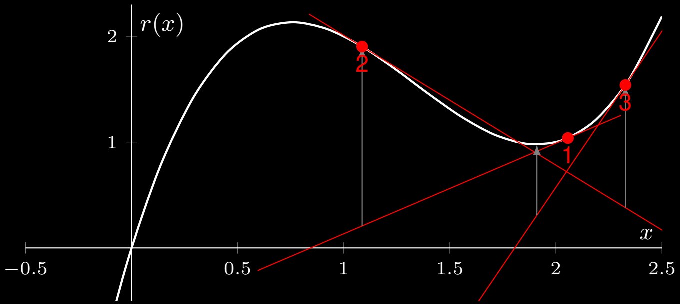

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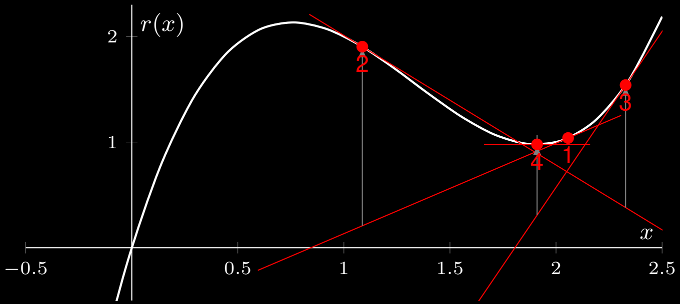

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Newton's method for equality constrained problems

Let's now try to apply Newton's method to the KKT conditions (89), observing that the unknowns are x, λ. We want to find the zero of the function

$$r(x, \lambda) = \begin{bmatrix} \nabla_x \mathcal{L}(x, \lambda) \\ h(x) \end{bmatrix}.$$

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which, by using $\mathcal{L}(x,\lambda) = f(x) + \lambda^\top h(x)$, can be simplified into

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and finally a simple re-organization gives

$$\underbrace{\begin{bmatrix} \nabla_x^2 \mathcal{L}(x,\lambda) & \nabla h(x) \\ \nabla h(x)^\top & 0 \end{bmatrix}}_{\text{The KKT matrix}} \begin{bmatrix} \Delta x \\ \lambda^+ \end{bmatrix} = - \begin{bmatrix} \nabla f(x) \\ h(x) \end{bmatrix},$$

where $\lambda^+ = \lambda + \Delta\lambda$ is the new iterate of the dual variable λ.

Newton's method for an equality constrained problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^\top \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 2 & 0 \end{bmatrix} x \\ \text{subject to} & x^\top x = 1 \end{array}$$

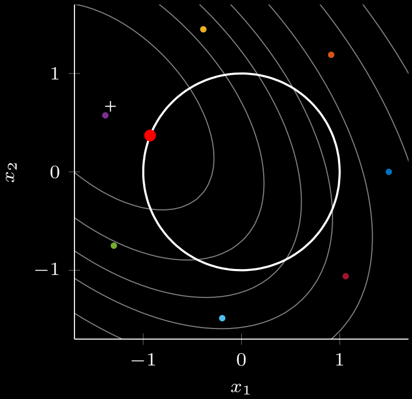

Figure 24: An illustration of the Newton method on an equality constrained problem. It can be seen that different initialisations require different number of steps until convergence.

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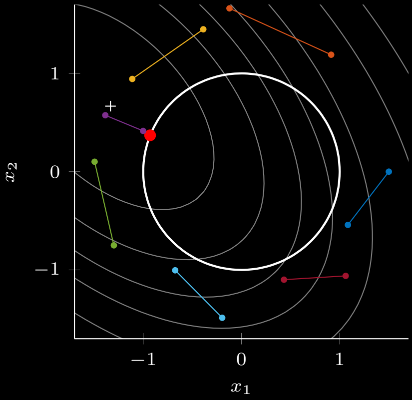

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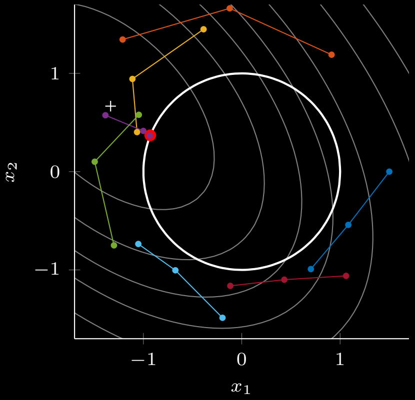

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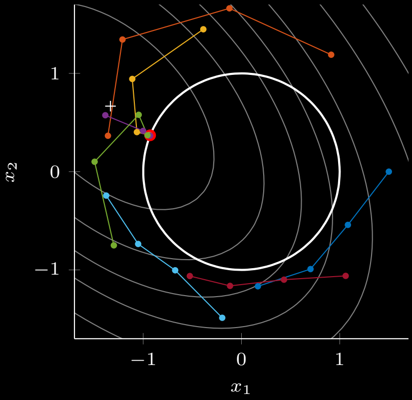

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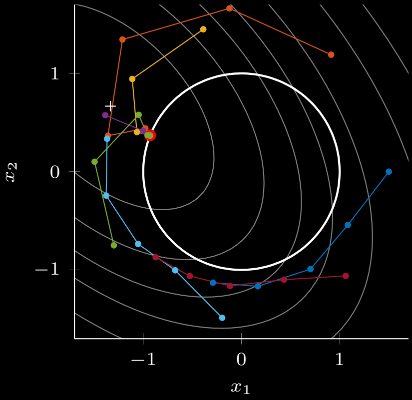

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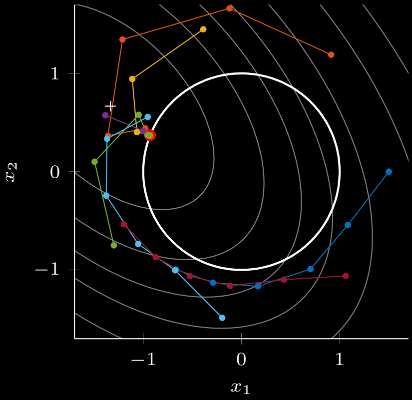

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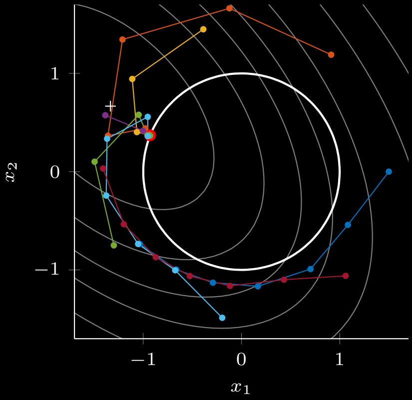

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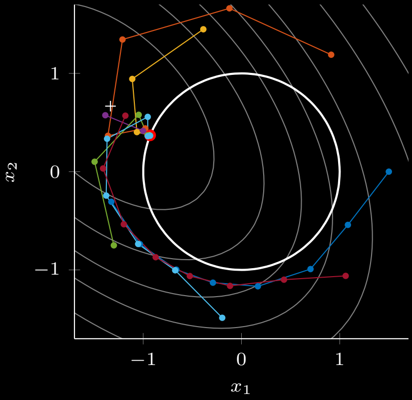

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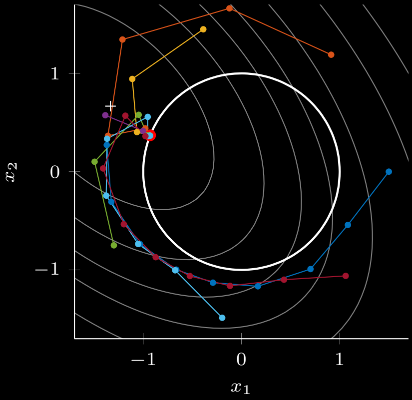

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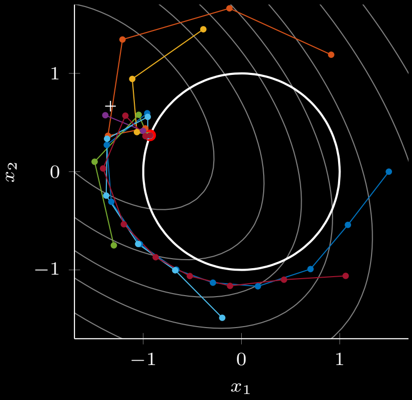

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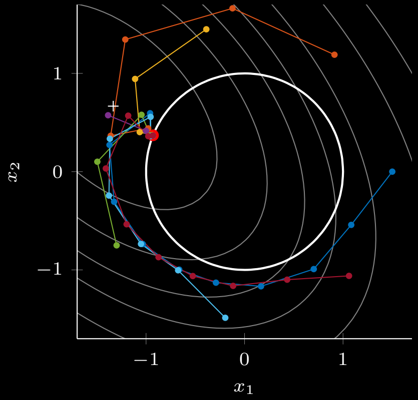

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Quadratic programming (QP)

$$\begin{aligned} \text{minimize} \quad & f(x) = \frac{1}{2}x^\top Qx + p^\top x, \quad Q \succ 0 && (72) \\ \text{subject to} \quad & Ax = b, \quad A \in \mathbb{R}^{p\times n}. && (73) \end{aligned}$$

The Lagrangian is

$$\mathcal{L}(x,\lambda) = \frac{1}{2}x^\top Qx + p^\top x + \lambda^\top (Ax - b).$$

and the Newton's method gives

$$\begin{bmatrix} Q & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \lambda^+ \end{bmatrix} = - \begin{bmatrix} Qx + p \\ Ax - b \end{bmatrix} \quad \Leftrightarrow$$

$$\begin{bmatrix} Q & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} x^+ \\ \lambda^+ \end{bmatrix} = \begin{bmatrix} -p \\ b \end{bmatrix}. \qquad (74)$$

Solutions to QP special cases

- Unconstrained case

$$\text{minimize} \quad f(x) = \frac{1}{2}x^\top Qx + p^\top x, \quad Q \succ 0$$

has a solution

$$Qx^* + p = 0.$$

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QP with inequality constraints

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}x^\top Q x + p^\top x, \quad Q \succeq 0 \\ \text{subject to} \quad & Gx \leq h \\ & Ax = b. \end{aligned} \tag{75}$$

For easy reference, we repeat the KKT conditions (84a)-(84d) for this special case

$$Qx^* + p + G^\top \mu^* + A^\top \lambda^* = 0 \tag{76}$$

$$\mu^* \geq 0 \tag{77}$$

$$Gx^* - h \leq 0, \quad Ax^* - b = 0 \tag{78}$$

$$\mu_i^*(g_i^\top x^* - h_i) = 0, \quad i = 1, \dots, m. \tag{79}$$

Active set method

Assume that a feasible point is known with specific active constraints $\mathbb{A}$. With this $\mathbb{A}$, the system of equations

$$\begin{bmatrix} Q & A^\top & G_{\mathbb{A}}^\top \\ A & 0 & 0 \\ G_{\mathbb{A}} & 0 & 0 \end{bmatrix} \begin{bmatrix} x^+ \\ \lambda^+ \\ \mu_{\mathbb{A}}^+ \end{bmatrix} = \begin{bmatrix} -p \\ b \\ h_{\mathbb{A}} \end{bmatrix}$$

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can be solved. Once this is done, there are two possible outcomes:

1. If the new point is feasible with respect to the (previously inactive) inequality constraints, we need to test if we are at the optimum. This is done by checking the Lagrange multipliers corresponding to the active set; they should all be nonnegative at the optimum. If this is not the case, the objective function can be further reduced by e.g. removing the constraint with the most negative multiplier from the active set, and the procedure is repeated.

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2. If the new point is not feasible, then the stepsize is reduced so that the new point becomes (just) feasible. This happens at the intersection with one of the previously inactive constraints. This is now added to the active set and the procedure is repeated.

Active set method

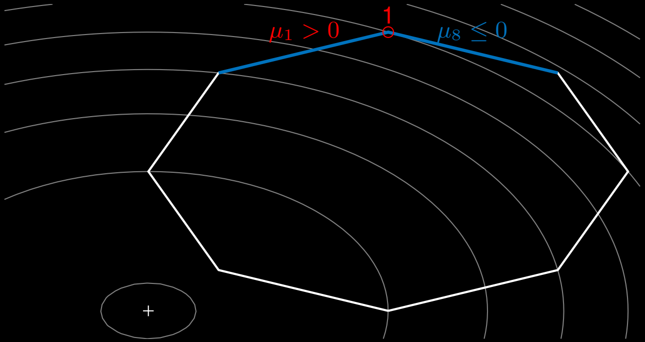
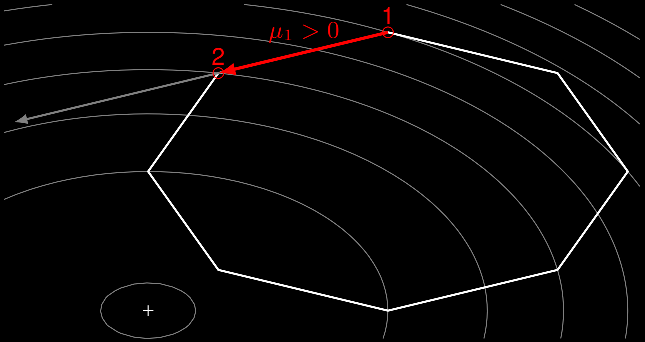


Figure 25: Illustration of active set method with two decision varaibles and initialised with two active constraints. The feasible region is within the octagon, the contour lines indicate the level sets of the cost function, the plus indicates the unconstrained optimum and the filled circle indicates the optimal solution.

Active set method



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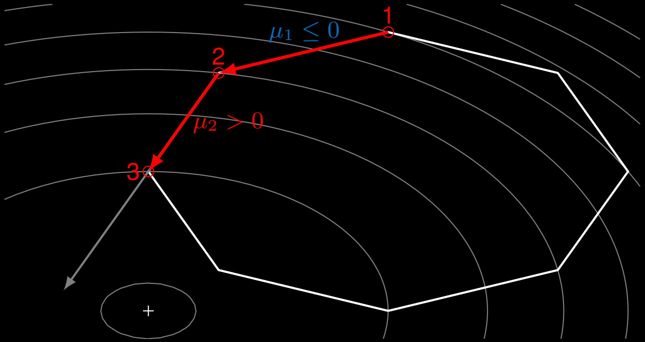
Active set method



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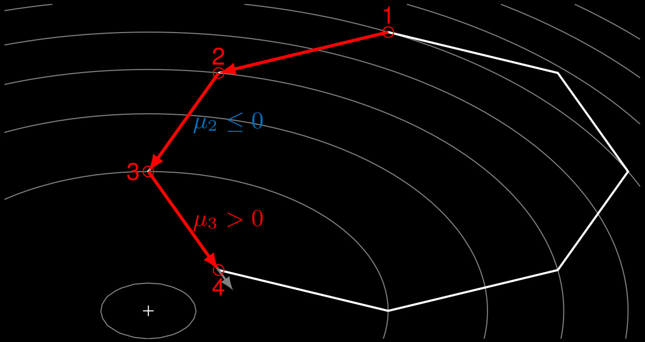
Active set method



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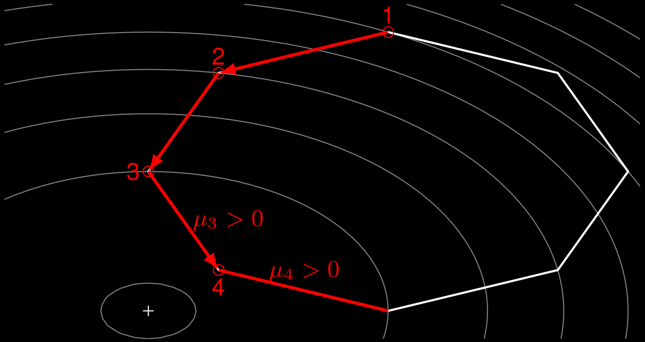

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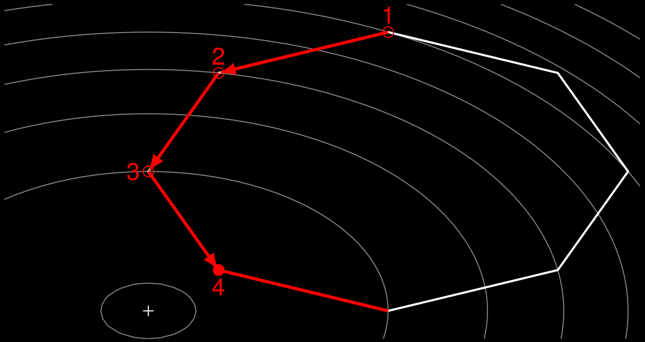

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Active set method 2

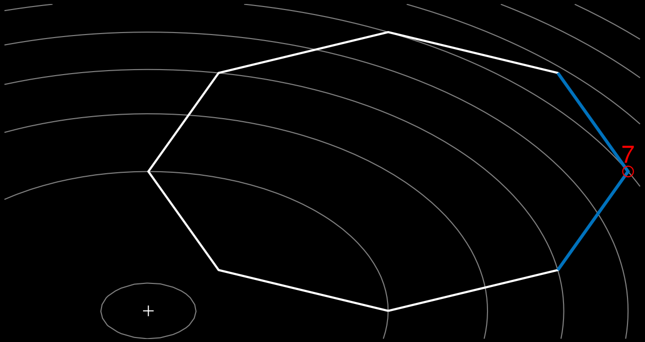
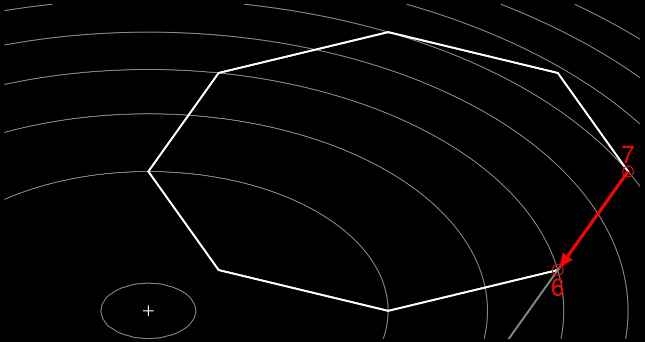


Figure 26: Illustration of active set method with two decision varaibles and initialised with two active constraints. The feasible region is within the octagon, the contour lines indicate the level sets of the cost function, the plus indicates the unconstrained optimum and the filled circle indicates the optimal solution.

Active set method 2



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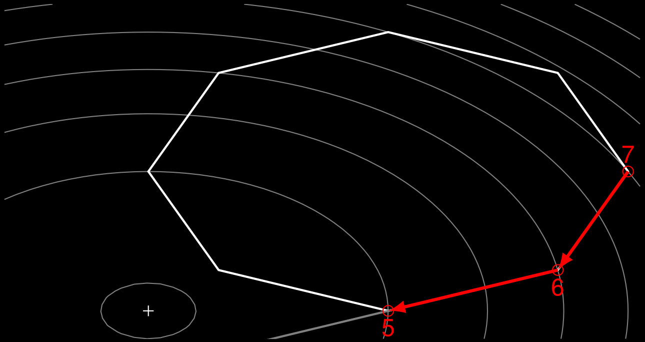

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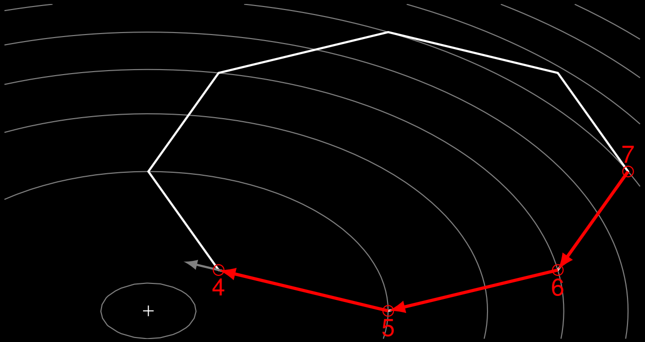

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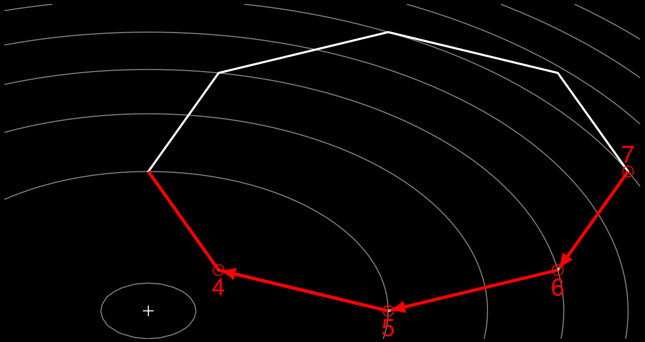

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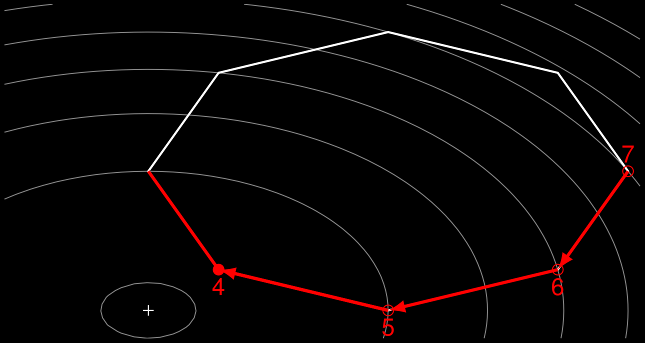

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Active set method 3

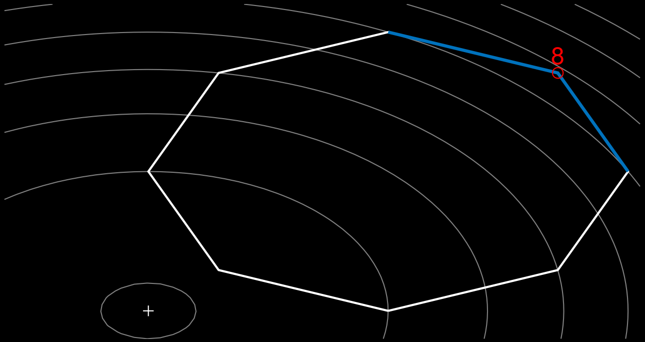

Figure 27: Illustration of active set method with two decision varaibles and initialised with two active constraints. The feasible region is within the octagon, the contour lines indicate the level sets of the cost function, the plus indicates the unconstrained optimum and the filled circle indicates the optimal solution.

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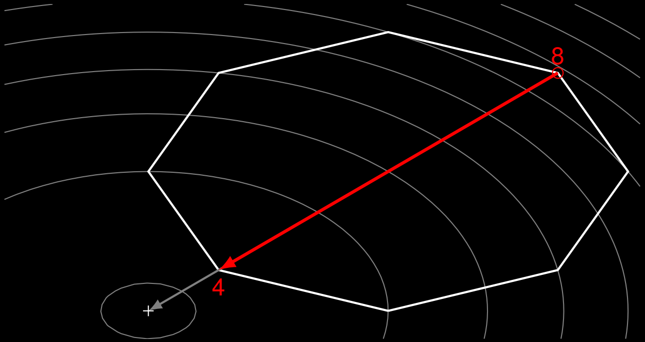

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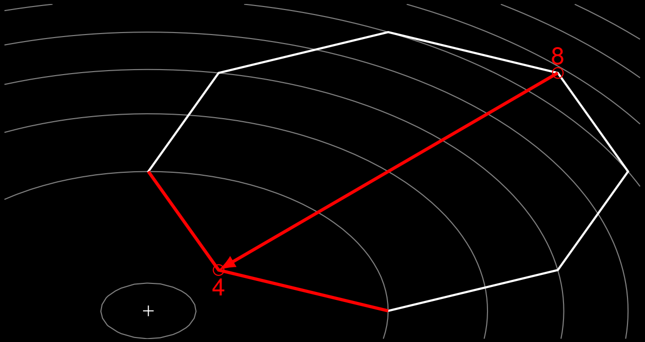

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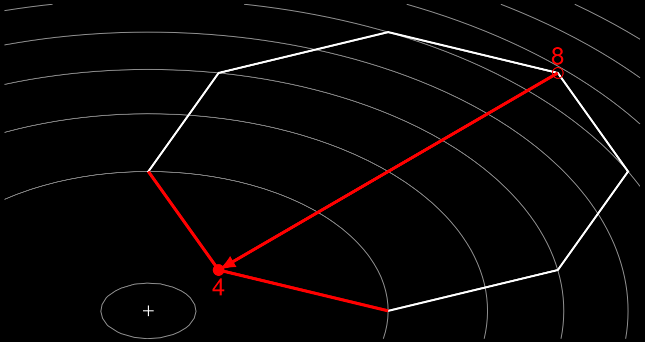

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Active set method 4

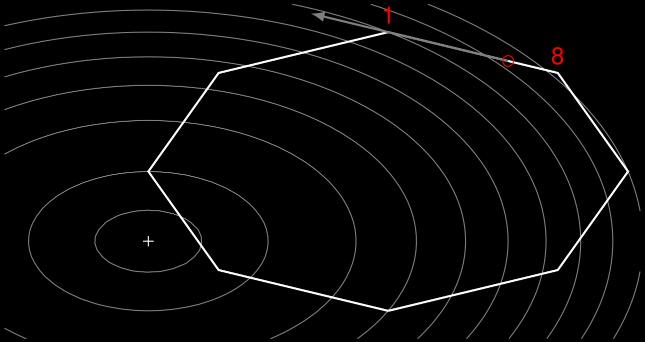
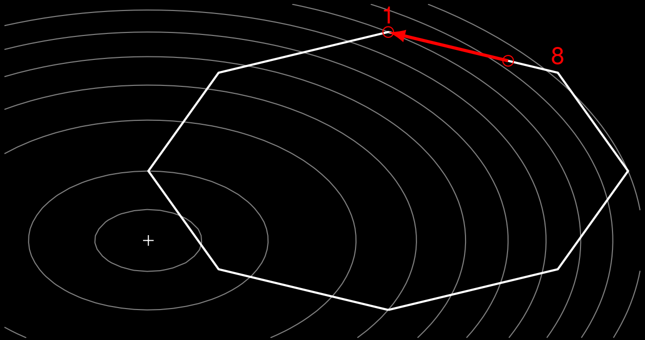


Figure 28: Illustration of active set method with two decision varaibles and initialised with one active constraint. The feasible region is within the octagon, the contour lines indicate the level sets of the cost function, the plus indicates the unconstrained optimum and the filled circle indicates the optimal solution.

Active set method 4



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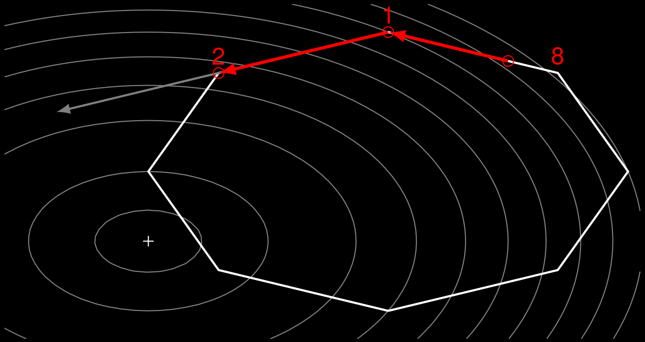

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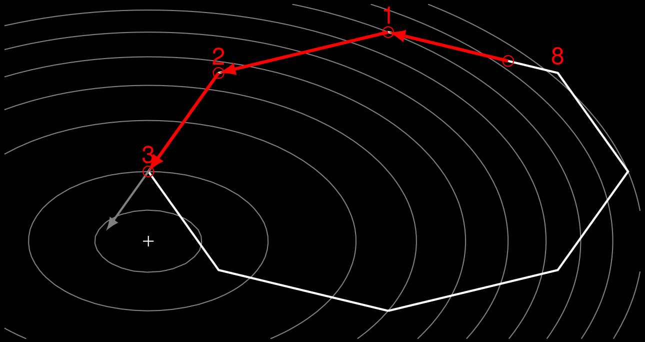

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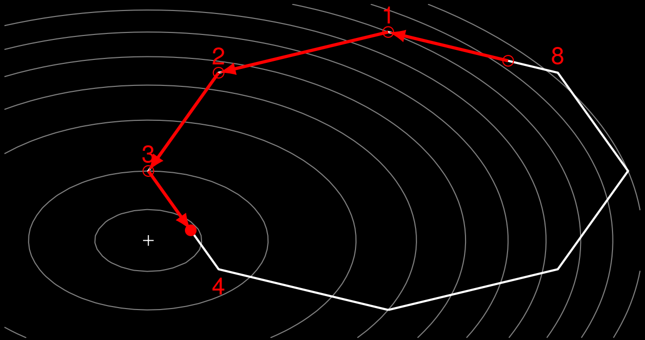

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Active set method 5

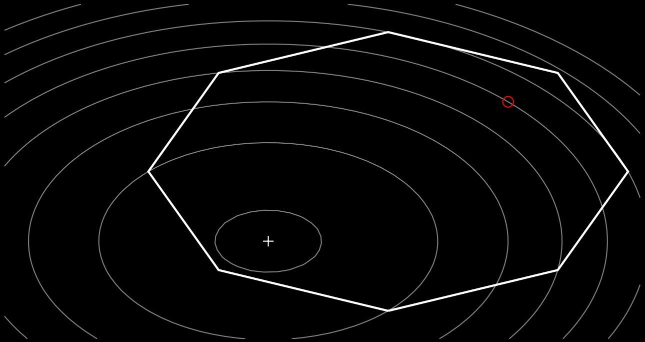

Figure 29: Illustration of active set method with two decision varaibles and initialised with inactive constraints. The feasible region is within the octagon, the contour lines indicate the level sets of the cost function, the plus indicates the unconstrained optimum and the filled circle indicates the optimal solution.

Active set method 5

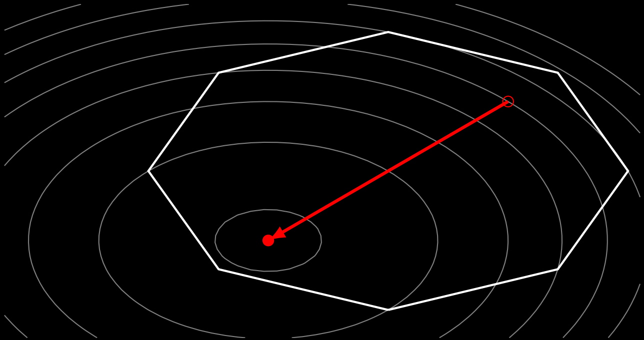

Figure 29: Illustration of active set method with two decision varaibles and initialised with inactive constraints. The feasible region is within the octagon, the contour lines indicate the level sets of the cost function, the plus indicates the unconstrained optimum and the filled circle indicates the optimal solution.

Interior point method: logarithmic barrier formulation

The QP problem

$$\begin{aligned} \text{minimize} \quad & f(x) = \frac{1}{2}x^\top Q x + p^\top x \\ \text{subject to} \quad & Gx \leq h \\ & Ax = b \end{aligned}$$

can be approximated by the following problem

$$\begin{aligned} \text{minimize} \quad & f_\tau(x) = f(x) - \tau \sum_{i=1}^{m} \log(h_i - g_i^\top x) \quad (\tau > 0) \\ \text{subject to} \quad & Ax = b \end{aligned}$$

where $g_i^\top$ is the ith row of G, h_i is the ith element of h.

Interior point method: logarithmic barrier formulation

The QP problem

$$\begin{aligned} \text{minimize} \quad & f(x) = \frac{1}{2}x^\top Q x + p^\top x \\ \text{subject to} \quad & Gx \leq h \\ & Ax = b \end{aligned}$$

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where $g_i^\top$ is the ith row of G, h_i is the ith element of h.
The *convex* function

$$\phi_\tau(x) = -\tau \sum_{i=1}^{m} \log(h_i - g_i^\top x)$$

is called the *logarithmic barrier* for the original QP problem.

Interior point barrier method

$$\begin{aligned} \text{minimize} \quad & f(x) = \frac{x^2}{2} - 2x \\ \text{subject to} \quad & -1 \le x \le 1 \end{aligned}$$

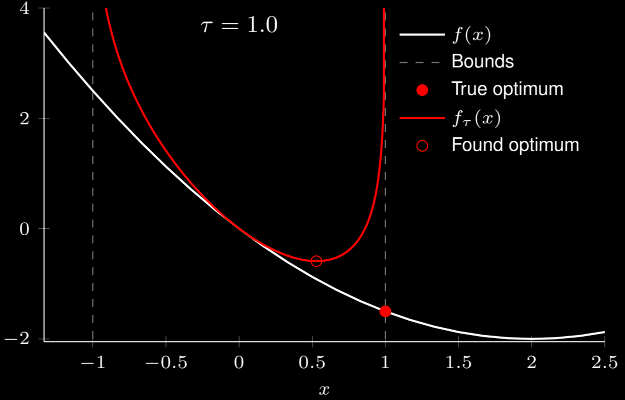
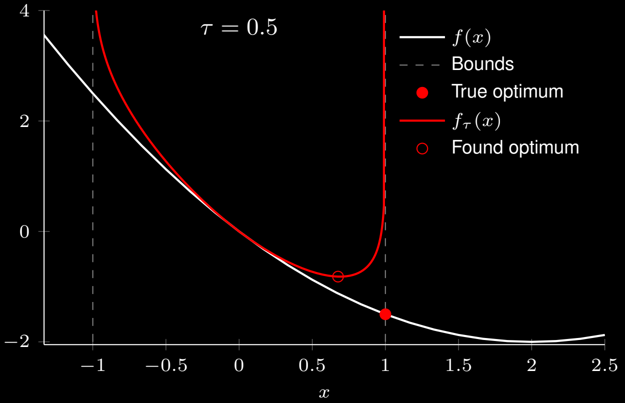


Figure 30: Illustration of the interior point barrier method.

Interior point barrier method

$$\text{minimize} \quad f(x) = \frac{x^2}{2} - 2x$$
$$\text{subject to} \quad -1 \leq x \leq 1$$



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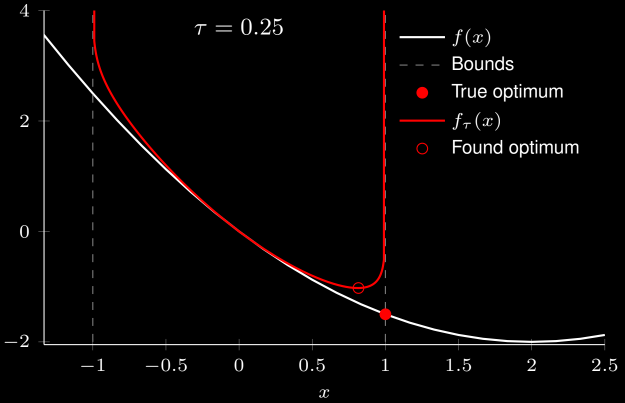

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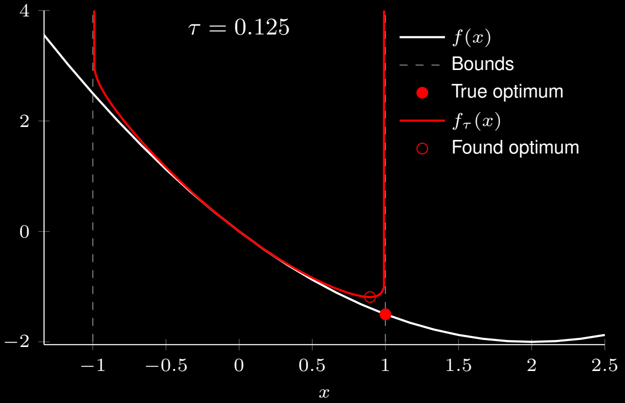
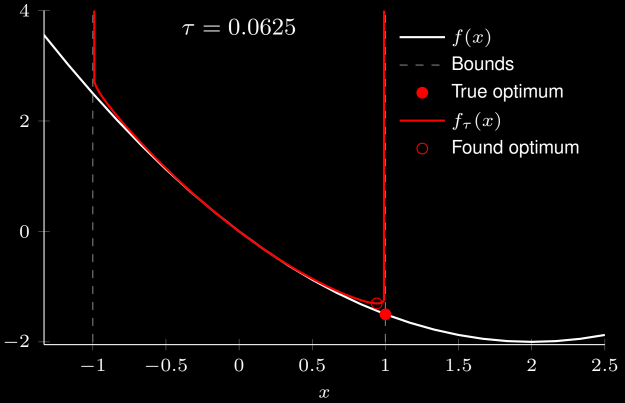


Figure 30: Illustration of the interior point barrier method.

Interior point barrier method

$$
\begin{aligned}
\text{minimize} \quad & f(x) = \frac{x^2}{2} - 2x \\
\text{subject to} \quad & -1 \leq x \leq 1
\end{aligned}
$$



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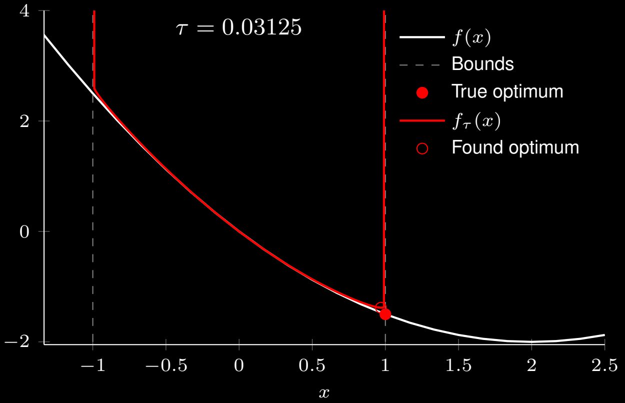

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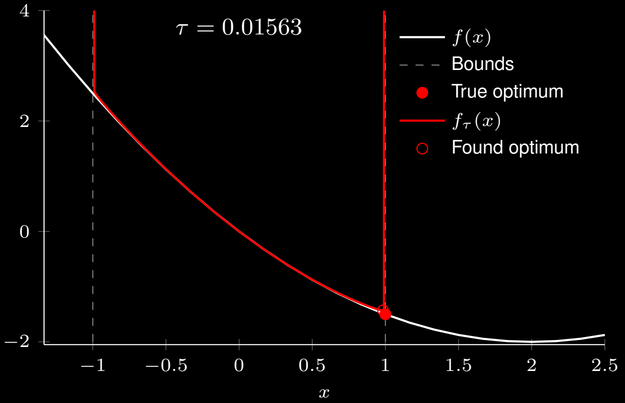

Figure 30: Illustration of the interior point barrier method.

KKT conditions for a barrier QP

$$
\begin{aligned}
&Qx + p + \tau \sum_{i=1}^{m} \frac{1}{h_i - g_i^\top x} g_i + A^\top \lambda = 0 \\
&Ax - b = 0
\end{aligned}
$$

which are valid only for *interior points*, i.e. those x satisfying $h_i - g_i^\top x > 0, \forall i$.

KKT conditions for a barrier QP

$$
\begin{aligned}
&Qx + p + \tau \sum_{i=1}^{m} \frac{1}{h_i - g_i^\top x} g_i + A^\top \lambda = 0 \\
&Ax - b = 0
\end{aligned}
$$

which are valid only for *interior points*, i.e. those x satisfying $h_i - g_i^\top x > 0, \forall i$.
By defining $\mu_i = \tau/(h_i - g_i^\top x)$, the KKT conditions for the barrier method can be rewritten as

$$
\begin{aligned}
&Qx + p + \sum_{i=1}^{m} \mu_i g_i + A^\top \lambda = 0 \\
&Ax - b = 0 \\
&\mu_i (h_i - g_i^\top x) = \tau
\end{aligned}
$$

which, together with the conditions $h_i - g_i^\top x > 0$ and $\mu_i > 0$, can be seen as a version of the original KKT conditions (76)-(79), where the complementary slackness conditions have been *smoothed*.

A primal-dual interior point method

The QP problem

$$\begin{aligned} \text{minimize} \quad & f(x) = \frac{1}{2}x^\top Q x + p^\top x \\ \text{subject to} \quad & Gx \leq h \\ & Ax = b \end{aligned}$$

is characterized by the approximated (smoothed) KKT conditions

$$\begin{aligned} & Qx + p + G^\top \mu + A^\top \lambda = 0 \\ & Ax - b = 0 \\ & Gx - h + s = 0 \\ & \mu_i s_i = \tau \\ & s > 0, \quad \mu > 0. \end{aligned}$$

A primal-dual interior point method

The QP problem

$$\begin{aligned} \text{minimize} \quad & f(x) = \frac{1}{2}x^\top Qx + p^\top x \\ \text{subject to} \quad & Gx \leq h \\ & Ax = b \end{aligned}$$

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Applying Newton's method on the equalities gives the Newton step

$$\begin{bmatrix} Q & A^\top & G^\top & 0 \\ A & 0 & 0 & 0 \\ G & 0 & 0 & I \\ 0 & 0 & \operatorname{diag}(s) & \operatorname{diag}(\mu) \end{bmatrix} \begin{bmatrix} x^+ \\ \lambda^+ \\ \mu^+ \\ s^+ \end{bmatrix} = \begin{bmatrix} -p \\ b \\ h \\ \operatorname{diag}(s)\mu + \tau \end{bmatrix}.$$

Backtracking to secure $s > 0$ and $\mu > 0$ is simple!

Newton's method for an inequality constrained problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^\top \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 2 & 0 \end{bmatrix} x \\ \text{subject to} & x^\top x \leq 1 \end{array}$$

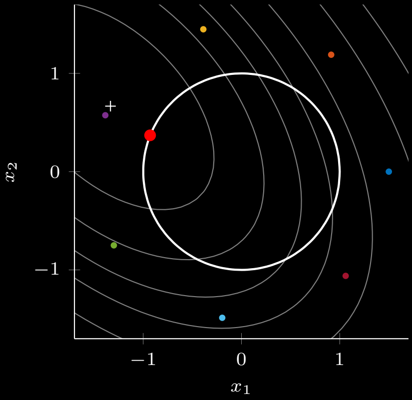
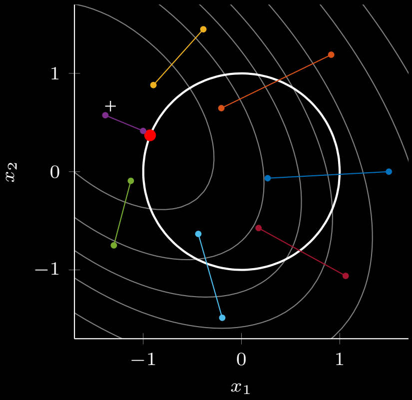


Figure 31: An illustration of the Newton method on an inequality constrained problem.

Newton's method for an inequality constrained problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^\top \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 2 & 0 \end{bmatrix} x \\ \text{subject to} & x^\top x \leq 1 \end{array}$$



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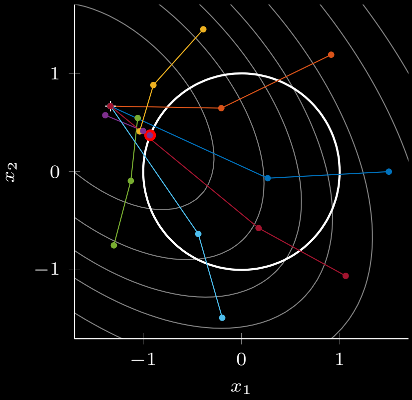

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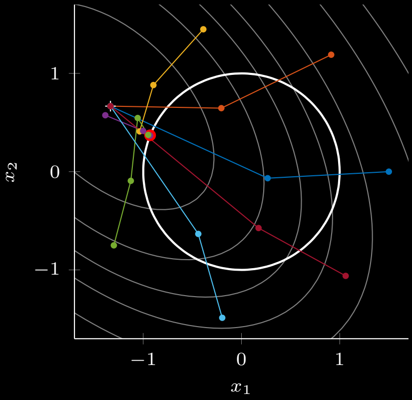

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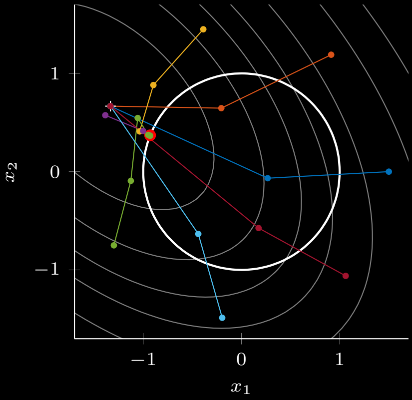

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References

[1] J.B. Rawlings, D.Q. Mayne, and M.M. Diehl. *Model Predictive Control: Theory, Computation, and Design, 2nd edition*. Nob Hill Publishing 2017.
Available online at `https://sites.engineering.ucsb.edu/~jbraw/mpc`

[2] G. Goodwin, M.M. Seron, and J.A. De Don·. *Constrained Control and Estimation*. Springer 2004. Available online via Chalmers Library.

[3] F Borrelli, A. Bemporad, and M. Morari. *Predictive Control for Linear and Hybrid Systems*. Available online at http://www.mpc.berkeley.edu/mpc-course-material

[4] J. Maciejowski. *Predictive Control with Constraints*. Prentice Hall 2002.

[5] S. Boyd and L. Vandenberghe. *Convex optimisation*. Cambridge University Press 2004.

[6] Diehl, M. *Real-Time Optimization for Large Scale Nonlinear Processes*. PhD thesis, University of Heidelberg, 2001.

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