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SSY281 - MODEL PREDICTIVE CONTROL

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Lecture 10: Stability

Goals for today:

- To understand and be able to use Lyapunov functions to study simple stability problems
- To understand the fundamental ingredients in establishing stability for receding horizon controllers
- To formulate and verify the stability conditions of constrained linear quadratic MPC

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Learning objectives:

- Describe basic properties of MPC controllers and analyse algorithmic details on very simple examples

Stability definitions

Consider the system

$$x^+ = f(x), \quad f(0) = 0. \tag{89}$$

Definition (Local stability)

The origin is *locally stable* for the system (89) if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|x(0)| < \delta$ implies $|x(k)| < \epsilon$ for all $k \geq 0$.

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Remark

If the origin is attractive only for initial states within a certain set, the definitions can be modified and the properties then hold with a region of attraction that is not any longer the entire $\mathbb{R}^n$.

Lyapunov function

Definition (Lyapunov function)

A function $V : \mathbb{R}^n \to \mathbb{R}_+$ is said to be a Lyapunov function for the system (89) if there are functions $\alpha_i \in \mathcal{K}_\infty, i = 1, 2$ and a positive definite function α_3 such that for any $x \in \mathbb{R}^n$,

$$V(x) \geq \alpha_1(|x|)$$
$$V(x) \leq \alpha_2(|x|)$$
$$V(f(x)) - V(x) \leq -\alpha_3(|x|).$$

Remark

A function belongs to $\mathcal{K}_\infty$ if it is nonnegative, continuous, zero at zero, strictly increasing and unbounded. A positive definite function is continuous and positive everywhere except at the origin.

Lyapunov's theorem

Proposition (Lyapunov's theorem)

Suppose $V(\cdot)$ is a Lyapunov function for the system

$$x^+ = f(x), \quad f(0) = 0.$$

Then the origin is globally asymptotically stable.

Basic MPC equations

Cost function:

$$V_N(x_0, u(0\!:\!N-1)) = \sum_{i=0}^{N-1} l(x(i), u(i)) + V_f(x(N)). \tag{90}$$

Optimal cost-to-go:

$$V_N^*(x_0) = \min_{u(0:N-1)} \{V_N(x_0, u(0\!:\!N-1)) \mid u(0\!:\!N-1) \in \mathcal{U}_N(x_0)\}.$$

Constraints:

$$x^+ = f(x,u), \quad x(0) = x_0 \tag{91}$$

$$x(k) \in \mathbb{X}, \quad u(k) \in \mathbb{U}, \quad \text{for all } k \in (0, N) \tag{92}$$

$$x(N) \in \mathbb{X}_f \subseteq \mathbb{X}. \tag{93}$$

Feasible control sequences and initial states:

$$u(0:N-1) \in \mathcal{U}_N(x_0) \tag{94}$$

$$\mathcal{X}_N = \{x_0 \in \mathbb{X} \mid \mathcal{U}_N(x_0) \neq \emptyset\}. \tag{95}$$

Optimal control and state sequences:

$$\begin{aligned} u^*(0:N-1;x_0) &= \{u^*(0;x_0), u^*(1;x_0), \dots, u^*(N-1;x_0)\} \\ x^*(0:N;x_0) &= \{x^*(0;x_0), x^*(1;x_0), \dots, x^*(N;x_0)\}. \end{aligned} \tag{96}$$

Existence of solution

Proposition (Existence of solution)

With the assumptions above, the following holds:

(a) *The function* V_N *is continuous on* $\mathcal{X}_N \times \mathcal{U}_N$.

(b) *For each* $x \in \mathcal{X}_N$, *the control constraint set* $\mathcal{U}_N(x)$ *is closed and bounded.*

(c) *For each* $x \in \mathcal{X}_N$, *a solution to the optimal control problem exists.*

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(c) *For each $x \in \mathcal{X}_N$, a solution to the optimal control problem exists.*

Remark

Note that nothing is said about the properties of the value function $V_N^*(x)$ or the control law $\kappa_N(x) = u^*(0; x)$. In fact, these may both be discontinuous. However, if there are no state constraints (i.e. $\mathbb{X} = \mathbb{X}_f = \mathbb{R}^n$) or if the system is linear and the constraint sets are all polyhedral, then the value function is continuous.

Equivalence between infinite and finite horizon LQ

The infinite horizon LQ problem is based on the criterion in equation (29),

$$V(x(0), u(0\!:\!\infty)) = \sum_{i=0}^{\infty} \left(x^\top(i) Q x(i) + u^\top(i) R u(i) \right).$$

If the sum is split into two terms we get

$$V(x(0), u(0\!:\!\infty)) = \sum_{i=0}^{N-1} \left(x^\top(i) Q x(i) + u^\top(i) R u(i) \right) + \sum_{i=N}^{\infty} \left(x^\top(i) Q x(i) + u^\top(i) R u(i) \right).$$

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By minimising the *finite time* horizon criterion

$$V(x(0), u(0\!:\!N-1)) = \sum_{i=0}^{N-1} \left(x^\top(i) Q x(i) + u^\top(i) R u(i) \right) + x^\top(N) P x(N),$$

we will in fact generate exactly the same minimising control sequence $u^*(0\!:\!N-1)$ as the infinite horizon formulation leads to.

Basic stability assumption

We proceed with the following basic assumption:

Assumption (Basic stability assumption)

The terminal set $\mathbb{X}_f$ is control invariant and the following inequality holds:

$$\min_{u \in \mathbb{U}} \{V_f(f(x,u)) + l(x,u) \mid f(x,u) \in \mathbb{X}_f\} \leq V_f(x), \quad \forall x \in \mathbb{X}_f.$$

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Remark

Recall that control invariance of $\mathbb{X}_f$ means that there exists a $u \in \mathbb{U}$ such that $f(x,u) \in \mathbb{X}_f$, see Definition 9.4. The implication of this, together with the properties of $\mathbb{U}$, is that the minimum in the assumption above exists.

Decay of the value function

Lemma (Value function decrease)

Assume that Assumption 1.3 holds. Then the optimal cost or value function fulfills the following inequality for all $x \in \mathcal{X}_N$*:*

$$V_N^*(f(x, \kappa_N(x))) \leq V_N^*(x) - l(x, \kappa_N(x)).$$

Proof of Lemma 16

Let x be any point in $\mathcal{X}_N$ with $V_N^*(x) = V_N(x, u^*(0\!:\!N-1; x))$. The corresponding optimal control and state sequences are as in equation (96):

$$
\begin{aligned}
u^*(0\!:\!N-1; x) &= \{u^*(0; x), u^*(1; x), \dots, u^*(N-1; x)\} \\
x^*(0\!:\!N; x) &= \{x^*(0; x), x^*(1; x), \dots, x^*(N; x)\},
\end{aligned}
$$

where $u^*(0; x) = \kappa_N(x)$, $x^*(0; x) = x$ and the successor state is $x^+ = x^*(1; x) = f(x, \kappa_N(x))$.

Proof of Lemma 16

Let x be any point in $\mathcal{X}_N$ with $V_N^*(x) = V_N(x, u^*(0\!:\!N-1; x))$. The corresponding optimal control and state sequences are as in equation (96):

$$u^*(0\!:\!N-1; x) = \{u^*(0; x), u^*(1; x), \ldots, u^*(N-1; x)\}$$
$$x^*(0\!:\!N; x) = \{x^*(0; x), x^*(1; x), \ldots, x^*(N; x)\},$$

where $u^*(0; x) = \kappa_N(x)$, $x^*(0; x) = x$ and the successor state is $x^+ = x^*(1; x) = f(x, \kappa_N(x))$. In order to compare $V_N^*(x)$ with $V_N^*(x^+)$, we note that

$$V_N^*(x^+) \le V_N(x^+, \tilde{u}(0\!:\!N-1)) \tag{97}$$

$$\tilde{u}(0\!:\!N-1) = \{u^*(1; x), \ldots, u^*(N-1; x), u\}$$

and the state sequence resulting from $\tilde{u}(0\!:\!N-1)$ is

$$\tilde{x} = \{x^*(1; x), \ldots, x^*(N; x), f(x^*(N; x), u)\}.$$

We have

$$V_N(x^+, \tilde{u}(0\!:\!N-1)) = V_N^*(x) - l(x, \kappa_N(x)) - V_f(x^*(N;x)) \\ + l(x^*(N;x), u) + V_f(f(x^*(N;x), u)) \leq V_N^*(x) - l(x, \kappa_N(x))$$

where the inequality follows from the fact that u can be chosen according to Assumption 1.3. Combining this result with equation (97) completes the proof.

MPC stability

Theorem (MPC stability)

Assume that Assumption 1.3 holds and that the stage cost $l(\cdot)$ and the terminal cost $V_f(\cdot)$ satisfy

$$
\begin{aligned}
l(x,u) &\geq \alpha_1(|x|) \quad \forall x \in \mathcal{X}_N, u \in \mathbb{U} \\
V_f(x) &\leq \alpha_2(|x|) \quad \forall x \in \mathbb{X}_f
\end{aligned}
$$

with $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. Further, assume that $\mathbb{X}_f$ contains the origin in its interior. Then the origin is asymptotically stable with a region of attraction $\mathcal{X}_N$ for the system $x^+ = f(x, \kappa_N(x))$.

Stability of constrained linear quadratic MPC

The system is now described by the state equation

$$x^+ = Ax + Bu$$

with (A, B) controllable and the stage cost is

$$l(x, u) = x^\top Qx + u^\top Ru$$

with $Q, R \succ 0$. As before, the constraint sets $\mathbb{X}$ and $\mathbb{U}$ are polyhedral, i.e. described by linear inequalities.

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with $Q, R \succ 0$. As before, the constraint sets $\mathbb{X}$ and $\mathbb{U}$ are polyhedral, i.e. described by linear inequalities.
Choose the terminal cost as the value function for the *unconstrained* LQ problem, i.e.

$$V_f(x) = V_\infty^{\text{uc}}(x) = x^\top Px,$$

where P is the solution of the algebraic Riccati equation. The value function satisfies the equation

$$V_\infty^{\text{uc}}(x) = \min_u \{x^\top Qx + u^\top Ru + V_\infty^{\text{uc}}(x^+)\} = x^\top Qx + (Kx)^\top R(Kx) + V_\infty^{\text{uc}}(Ax + BKx)$$

which implies that

$$V_f((A + BK)x) + x^\top Qx + (Kx)^\top R(Kx) = V_f(x).$$

This means that the chosen V_f satisfies the inequality of Assumption 1.3, *if* we can ensure that constraints are not active in $\mathbb{X}_f$. This can indeed be guaranteed if we define $\mathbb{X}_f \subseteq \mathbb{X}$ to be the largest set fulfilling the following two conditions:

1. $x \in \mathbb{X}_f \Rightarrow Kx \in \mathbb{U}$ and
2. $x \in \mathbb{X}_f \Rightarrow (A + BK)^i x \in \mathbb{X}_f$ for all $i \geq 0$.

The set $\mathbb{X}_f$ thus defined is control invariant.

Stability of constrained linear quadratic MPC

Theorem (Stability of constrained linear quadratic MPC)

Consider the linear quadratic MPC with linear constraints applied to the controllable system $x^+ = Ax + Bu$ *and with positive definite matrices* Q *and* R*. Further assume that the terminal cost* V_f *is chosen as the value function of the corresponding unconstrained, infinite horizon LQ controller, and that the terminal constraint set* $\mathbb{X}_f$ *is chosen as described above. Then the origin is asymptotically stable with a region of attraction* $\mathcal{X}_N$ *for the controlled system* $x^+ = Ax + B\kappa_N(x)$*.*

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