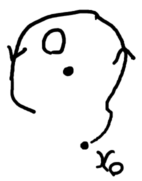
den 19 februari 2021 07:46

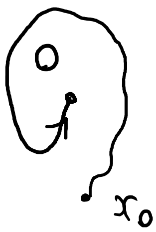
PSS 4 - Stability

Today: Exercise #1 ; Exercise 5.2

Summary:

plant: $x(k+1) = f(x(k))$, with $f(0) = 0$.

Stability? 

Asymptotic stability: 

Lyapunov functions: $V: \mathbb{R}^n \to \mathbb{R}^+$

V is a Lyapunov function if: (i) $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$
(ii) $V(f(x)) - V(x) \leq -\alpha_3(|x|)$

where α_1, α_2 are K^∞ functions and α_3 is a positive definite function.

(To simplify, think of (i) and (ii) as: (i) $V(x) > 0$, $x \neq 0$, and $V(0) = 0$.
(ii) $V(f(x)) - V(x) < 0$)

Prop 10.1: If $V(.)$ is a Lyapunov function, then the origin is globally asymptotically stable.

What about linear systems? $x(k+1) = Ax(k)$

Easy: stability $\Leftrightarrow$ eig(A) inside the unit circle.

Still, we can build some Lyapunov functions:

$V(x) = x^T S x$, with $S \succeq 0$.

Is $V(.)$ a Lyap function?

(i) OK

(ii) $V(x_{k+1}) - V(x_k) = -x^T\underbrace{(S - A^T S A)}_{Q}x = -x^T Q x$

$V(\cdot)$ is Lyap $\Leftrightarrow Q \succ 0$.

Lemma 2.7: A stable $\Leftrightarrow \forall Q \succ 0, \exists!\, S \succ 0$ such that the discrete time Lyapunov equation holds: $S - A^T S A = Q$

Excercise #1:

Consider $x(k+1) = \begin{bmatrix} 0.5 & 1 \\ -0.1 & 0.2 \end{bmatrix} x(k)$

a) Find a Lyap function $V(x) = x^T S x$

hint: dlyap in Matlab.

first, A is asymptotically stable since eig$(A) = 0.35 \pm 0.28i$

Lemma 2.7 : pick $Q = I_2$, then solve Lyap equation.

$S = \begin{bmatrix} 1.29 & 0.59 \\ 0.59 & 2.63 \end{bmatrix} \succ 0$, so $V(x) = x^T S x$ is a Lyap function.

⚠ use A^T for A in dlyap (cf its doc).

b) Unconstrained ∞ horizon RHC:

$$V(u, x_0) = \sum_{i=0}^{\infty} \left(x(i)^T Q x(i) + u(i)^T R u(i) \right), \quad Q = I_2, \; R = 1$$

s.t. $x(k+1) = Ax(k) + Bu(k)$; $A = \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Show that $V(u, x_0)$ is a Lyap function for the closed-loop system.

Closed-loop control by solving DARE: (use Matlab)

$P = \begin{bmatrix} 13.66 & 7.99 \\ 7.99 & 39.44 \end{bmatrix}$ $\quad K = \begin{bmatrix} -0.78 & 5.07 \end{bmatrix}$

now use:

$V(u, x_0) = x_0^T P x_0$ since ∞ horizon control.

To verify that this Lyap candidate is a Lyap function:

(i) $P \succ 0$?

(ii) $V(u, x_1) - V(u, x_0) = -x_0^T Q x_0$, is $Q \succ 0$?

(i) $\mathrm{eig}(P) = \{11.4 ; 41.7\} \Rightarrow P \succ 0$

(ii) $V(u, x_1) - V(u, x_0) = x_1^T P x_1 - x_0^T P x_0$

$$= x_0^T (A + BK)^T P (A + BK) x_0 - x_0^T P x_0$$

$$= -x_0^T Q x_0$$

where $Q = P - (A+BK)^T P (A+BK) = \begin{bmatrix} 1.60 & -3.95 \\ -3.95 & 26.75 \end{bmatrix}$

$\text{eig}(Q) = \{1 ; 27{,}35\}$

(i) and (ii) hold: $V(u, x_0)$ is a Lyapunov function.

$\boxed{5.2}$

⊛ First, we prove this in the general case.

Here: $u(k+i) = 0, \ \forall i \geq M$.

assume: $Z(k) = x(k)$.

$$V(k) = \underbrace{\sum_{i=M}^{\infty} \|Z(k+i)\|_Q^2}_{\text{tail}} + \sum_{i=0}^{M-1} \left\{ \|Z(k+i)\|_Q^2 + \|\Delta u(k+i)\|_R^2 + \|u(k+i)\|_S^2 \right\}$$

$$\text{tail}: \sum_{i=M}^{\infty} \|Z(k+i)\|_Q^2 = x(k+M)^T \bar{Q}\, x(k+M).$$

How? $$\sum_{i=M}^{\infty} \cdots = x(k+M)^T Q\, x(k+M) + \overbrace{x(k+M+1)}^{= A x(k+M) + B\overset{=0}{u(k+M)}}{}^T Q\, x(k+M+1) + \ldots$$

$$= x(k+M)^T \underbrace{\left(\sum_{i=0}^{\infty} (A^T)^i Q A^i \right)}_{= \bar{Q}} x(k+M)$$

How do we get rid of the infinite sum?
Trick: use Lyap equation!

$$A^T \bar{Q} A = \sum_{i=0}^{\infty} (A^T)^{i+1} Q A^{i+1}$$

$\downarrow j = i+1$

∞

$$= \sum_{j=1}^{\infty} (A^T)^j Q A^j \qquad \downarrow j = i+1$$

$$\underbrace{\sum_{j=1}^{\infty} (A^T)^j Q A^j}_{\text{this is } \bar{Q} \text{ without its first term}}$$

$A^T \bar{Q} A = \bar{Q} - Q$ Lyap equation. $(S \leftrightarrow \bar{Q})$

Lemma 2.7: A stable $\Leftrightarrow \forall Q, \exists! \bar{Q}$.

We can now compute $\bar{Q}$ by solving Lyap equation.

$$\min V(k) = \sum_{i=0}^{M-1} \{ \|.\|^2 + \ldots \} + z(k+M)^T \bar{Q} z(k+M) \qquad (P)$$

$$\text{s.t. } x(k+1) = Ax(k) + Bu(k)$$

We have proved that $(P) \Leftrightarrow (P\text{-}I)$, if A stable.

(*) Second: assemble (P) as a QP with the given values.

$M = 2$ so optimization variables: $w = [x(k+1) \;\; x(k+2) \;\; u(k) \;\; u(k+1)]^T$

$$H = \begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & \bar{Q} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

solve: $(0.9^2 - 1)\bar{Q} = -1$ (from part 1)

so QP: $\min_{w} V(k) = w^T H w$ can be solved!

$\text{s.t. } A_{eq} w = b_{eq}$