Problem set 1

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Deadline for this assignment: February the 16th at noon. Please send your solutions to catena@chalmers.se.

Exercise 1 (10 points)

a) Let $S_{\mu\nu}$ be a set of coefficients and u^ν a contravariant vector. Show that if ω_μ, defined by

$$\omega_\mu = S_{\mu\nu} u^\nu \,, \tag{1}$$

is a covariant vector, then the coefficients $S_{\mu\nu}$ are the components of a covariant tensor (*3 points*).

b) Consider the constant ω and the coordinate transformation

$$\begin{aligned} t' &= t \\ x' &= x \cos \omega t + y \sin \omega t \\ y' &= -x \sin \omega t + y \cos \omega t \\ z' &= z \,. \end{aligned} \tag{2}$$

Furthermore, consider the contravariant vector u of components $u^\mu = \delta^\mu_t$ in the (t, x, y, z) coordinate system[1]. Find the components of u in the (t', x', y', z') coordinate system (*3 points*).

c) Consider the metric $g_{\mu\nu}$ with associated proper time interval in the (t, x, y, z) coordinate system,

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - [1 + \varphi(t, x)]\, \mathrm{d}y^2 - [1 - \varphi(t, x)]\, \mathrm{d}z^2 - 2\psi(t, x)\mathrm{d}y\mathrm{d}z \,. \tag{3}$$

Express $g_{\mu\nu}$ in the (u, v, y', z') coordinate system defined by

$$u = \frac{t - x}{\sqrt{2}} \,; \qquad v = \frac{t + x}{\sqrt{2}} \,; \qquad y' = y \,; \qquad z' = z \,. \tag{4}$$

(*4 points*).

Exercise 2 (10 points)

a) Given a constant ω, and the metric tensor $g_{\mu\nu}$ associated with the proper time interval

$$\mathrm{d}\tau^2 = \left[1 - \omega^2 (x^2 + y^2)\right] \mathrm{d}t^2 - 2\omega(x\mathrm{d}y - y\mathrm{d}x)dt - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \,, \tag{5}$$

find the components of the affine connection, $\Gamma^\mu_{\rho\sigma}$, in the coordinate system (t, x, y, z) (*4 points*).

[1] More explicitly, $u^\mu = (1, 0, 0, 0)$ in the (t, x, y, z) coordinate system.

b) Given a constant ω, show that the metric tensor $g_{\mu\nu}$ associated with the proper time interval

$$\mathrm{d}\tau^2 = \left[1 - \omega^2(x^2+y^2)\right]\mathrm{d}t^2 - 2\omega(x\mathrm{d}y - y\mathrm{d}x)dt - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2\,, \tag{6}$$

can be transformed in the Minkowski tensor by means of an appropriate coordinate transformation. Is there a gravitational field the spacetime region described by Eq. (6) (*6 points*).

Exercise 3 (10 points)

a) Prove the identity

$$\frac{\partial}{\partial x^\lambda} g^{\rho\sigma} = -g^{\rho\mu}g^{\sigma\nu}\frac{\partial}{\partial x^\lambda} g_{\mu\nu} \tag{7}$$

(*2 points*).

b) Given an arbitrary four-vector of contravariant components V^μ, show that

$$\left(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu\right)V^\rho = -R^\rho{}_{\sigma\mu\nu}V^\sigma\,, \tag{8}$$

where ∇_μ is a covariant derivative (*4 points*). Here we use Weinberg's definition of Riemann tensor, which might differ by an overall minus sign compared to other books (e.g. Carroll's book).

c) Given an antisymmetric tensor $F^{\mu\nu}$, show that

$$\left(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu\right)F^{\mu\nu} = 0\,, \tag{9}$$

(*4 points*).

Exercise 4 (10 points)

a) Prove the identity

$$\nabla_\rho R^\rho{}_{\sigma\mu\nu} = \nabla_\mu R_{\sigma\nu} - \nabla_\nu R_{\sigma\mu} \tag{10}$$

(*4 points*).

b) Let us now assume that the Ricci tensor, $R_{\mu\nu}$, and the metric tensor, $g_{\mu\nu}$, are proportional,

$$R_{\mu\nu} = \lambda g_{\mu\nu} \tag{11}$$

where λ is a constant. Show that:

- λ is necessarily equal to $R/4$, where R is the curvature scalar
- for each four-vector of covariant and contravariant components V_μ and V^μ, respec-

tively,

$$\left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) V^\mu = -\lambda V_\nu \,, \tag{12}$$

- and, finally,

$$\nabla_\rho \left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) V^\rho = -R^\rho{}_{\sigma\mu\nu} \nabla_\rho V^\sigma \,, \tag{13}$$

(*6 points*). In order to prove Eqs. (12) and (13), you might find useful Eq. (8) above (Ricci identity).