Problem set 2

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Deadline for this assignment: March the 3rd at 24:00. Please send your solutions to catena@chalmers.se.

Exercise 1 (10 points)

a) Show that the geodesic equation for a particle of mass m and four-momentum p^μ can be written as

$$m\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} = \frac{1}{2}\left(\frac{\partial g_{\nu\lambda}}{\partial x^\mu}\right) p^\lambda p^\nu \,, \tag{1}$$

where τ is the proper time. Use this result to prove that if the metric $g_{\mu\nu}$ does not depend on the coordinate $x^{\sigma *}$, then the component of the particle four-momentum $p_{\sigma *}$ is constant along the particle path (*3 points*).

b) Show that the geodesic equation for a particle of mass m and four-momentum p^μ can be written as

$$p^\mu \nabla_\mu p^\nu = 0 \,, \tag{2}$$

where ∇_μ is a covariant derivative. Use this result to prove that $\xi_\nu p^\nu$ is constant along the particle path[1], i.e.

$$\frac{\mathrm{d}(\xi_\nu \mathrm{p}^\nu)}{\mathrm{d}\tau} = p^\mu \nabla_\mu \left(\xi_\nu p^\nu\right) = 0 \,, \tag{3}$$

if and only if the four-vector ξ^μ in Eq. (3) is a Killing vector of the underlying metric tensor (*4 points*).

c) Let us now consider the metric tensor of line element

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2) \tag{4}$$

and the following notation: $x^0 = t$, $x^1 = r$, $x^2 = \vartheta$ and $x^3 = \varphi$. Show that the p_0 and p_3 components of a particle four-momentum are conserved along the particle trajectory, and that the vector fields $\eta^\mu = \delta^\mu_0$ and $\xi^\mu = \delta^\mu_3$ are Killing vectors of the given metric tensor (*3 points*).

Exercise 2 (10 points)

a) Let us consider the matter action

$$I_M = \int d^4x \sqrt{g}\, \mathscr{L}_M(x) \tag{5}$$

[1] Notice that for a scalar quantity, like $\xi_\nu p^\nu$, total derivative and covariant derivative along the particle path coincide.

where $\mathscr{L}_M(x)$ si the corresponding Lagrangian density. Show that the associated energy-momentum tensor, defined via

$$\delta I_M = \frac{1}{2}\int d^4x\sqrt{g}\,T^{\mu\nu}(x)\delta g_{\mu\nu}(x)\,, \tag{6}$$

can be written as

$$T^{\mu\nu} = 2\frac{\partial \mathscr{L}_M}{\partial g_{\mu\nu}} + \mathscr{L}_M g^{\mu\nu}\,, \tag{7}$$

(*4 points*).

b) Let $T^{\mu\nu}$ be the energy-momentum tensor of an electromagnetic field with four-vector potential A_μ:

$$T^{\mu\nu} = F^{\mu}{}_{\rho}F^{\nu\rho} - \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\,, \tag{8}$$

where

$$F_{\mu\nu} = \frac{\partial}{\partial x^\mu}A_\nu - \frac{\partial}{\partial x^\nu}A_\mu\,. \tag{9}$$

Show that if $T^{\mu\nu}$ is the only source term in Einstein equations, then these can be written as

$$R_{\mu\nu} = -8\pi G T_{\mu\nu}\,, \tag{10}$$

(*3 points*).

c) If $T_{\mu\nu}$ is the energy-momentum tensor of a perfect fluid, show that $T^{\mu\nu}T_{\mu\nu} = 0$ implies $T_{\mu\nu} = 0$ (*3 points*).

Exercise 3 (10 points)

a) Let us define $x^0 = t$, $x^1 = r$, $x^2 = \vartheta$ and $x^3 = \varphi$ and consider the metric tensor of line element

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2)\,, \tag{11}$$

where $f(r)$ is a differentiable function of r only. Let us also consider the four-vector velocity

$$u^\mu = e^{\psi(r,\vartheta)}\,[\delta^\mu_0 + \Omega(r,\vartheta)\,\delta^\mu_3] \tag{12}$$

where Ω and ψ are differentiable functions of r and ϑ only. By imposing

$$u_\mu u^\mu = -1, \tag{13}$$

express $e^{\psi(r,\vartheta)}$ in terms of $f(r)$ and $\Omega(r,\vartheta)$ (*2 points*).

b) Within the same assumptions at point a), show that if u^μ is tangent to a geodesic, namely

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\rho\sigma} u^\rho u^\sigma = 0\,, \tag{14}$$

then ψ and Ω are constant along the given geodesic and, furthermore, can be written in terms of $f(r)$ and its first derivative $f'(r)$, as long as $f'(r) \geq 0$ and $2f(r) - rf'(r) \geq 0$. Finally, briefly comment on the shape of the geodesic to which the four-vector u^μ is tangent (*6 points*).

c) In addition to the assumptions at point a), let us now consider the following form for $f(r)$:

$$f(r) = \left(1 - \frac{2MG}{r}\right) \tag{15}$$

where M is the mass of the underlying gravitational source and G Newton constant. Express Ω and $T \equiv 2\pi/\Omega$ as a function of G, M and the radial coordinate r, and then compare these expressions with the angular frequency and period of a keplerian orbit (*2 points*).

Exercise 4 (10 points)

a) Let us consider a test particle of mass m moving along a radial trajectory in the equatorial plane of a spacetime region described by the Schwarzshild solution to Einstein equation. Using the notation $x^0 = t$, $x^1 = r$, $x^2 = \vartheta$ and $x^3 = \varphi$, show that for this trajectory $p_\varphi = p_3 = p_\vartheta = p_2 = 0$, and $p_0 = p_t = -E$, where E is a constant of motion (*2 points*).

b) Using the relation between mass and four-momentum of the test particle at point a), namely

$$g_{\mu\nu} p^\mu p^\nu = -m^2\,, \tag{16}$$

and $p^1 = m\mathrm{d}r/\mathrm{d}\tau$, where τ is the proper time, show that

$$r^2 \left(1 - \frac{2MG}{r}\right) \left[E^2 - m^2 - \left(p^1\right)^2\right] = L^2 \tag{17}$$

where L is a second constant of motion. What is the value of L corresponding to a radial orbit? Use Eq. (17) to calculate the radial velocity, p^1/m, for both inward and outward orbits (*6 points*).

c) Show that in the Schwarzshild spacetime the surfaces

$$t + r + 2MG \ln\left|\frac{r}{2MG} - 1\right| = \text{constant} \tag{18}$$

are null hypersurfaces, i.e. the associated normal four-vector vector, n_μ, is a null vector; (*2 points*).