Problem set 3

Riccardo Catena

Deadline for this assignment: March the 10th at 24:00. Please submit your solutions via Canvas.

Exercise 1 (10 points)

a) Determine the critical value of the energy density, ρ_c, such that a universe described by the Robertson-Walker metric has zero curvature constant, k. Express ρ_c in g cm^{-3} (*2 points*).

b) Assuming $k = 0$, determine the time evolution of the scale factor, $R(t)$, in a universe with total energy density $\rho = \rho_M + \rho_\Lambda$. Here, ρ_M and ρ_Λ are the contributions to the total energy density ρ of a matter and a cosmological constant component, respectively (*4 points*).

c) Combine Friedmann equation with the continuity equation,

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} = \frac{1}{R^3(t)} \frac{\mathrm{d}}{\mathrm{d}t} \left\{ R^3(t) \left[p(t) + \rho(t) \right] \right\} \tag{1}$$

to obtain the tt-component of Einstein equations, $R_{tt} = -8\pi G S_{tt}$, in a Robertson-Walker universe. Here,

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\rho_{\ \rho} \,, \tag{2}$$

where $T_{\mu\nu}$ is the energy momentum tensor of a perfect fluid of pressure p and energy density ρ. Under what circumstances does the Universe undergo a stage of accelerated expansion, i.e. $\ddot{R} > 0$? Provide an example of perfect fluid that leads to an accelerating universe (*4 points*).

Exercise 2 (10 points)

a) Let us consider two infinitesimally close geodesics $x^\mu(\tau)$ and $x^\mu(\tau) + \delta x^\mu(\tau)$. Being infinitesimally close to each other, at leading order in δx^μ the two geodesics have the same tangent vector $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$. Write down the equation of geodesic deviation in terms of δx^μ, u^μ and the Riemann tensor. Here, τ is the proper time of an observer of four-velocity u^μ (*2 points*).

b) Let us now consider the Roberston-Walker metric with scale factor $R(t)$, $k = 0$, and line element,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + R^2(t) \left[\mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2 \right) \right] \,. \tag{3}$$

Let us also consider two infinitesimally close geodesics of tangent vector $u^\mu = \delta^\mu_t$ and displacement vector $\delta x^\mu = \delta^\mu_r$. Use the geodesic deviation equation to calculate the relative acceleration $D^2\delta x^\mu/D\tau^2$, where $D/D\tau$ denotes a covariant derivative along the geodesics. Express this relative acceleration as a function of the Hubble parameter $H \equiv \dot{R}/R$ (where a dot denotes a derivative with respect to t) and the deceleration parameter $q \equiv -R\ddot{R}/\dot{R}$ (*4 points*).

c) Assuming Eq. (2) as a source term for Einstein equations, express q as a function of p and ρ (*2 points*).

d) Assuming $H > 0$, under what conditions is the relative acceleration at point b) positive (*2 points*).

Exercise 3 (10 points)

a) Extract the infinitesimal area element of a sphere of comoving radius r from the Robertson-Walker metric (*2 points*).

b) Calculate the area of a sphere of comoving radius r in Robertson-Walker geometries with $k = -1, 0, +1$ (*2 points*).

c) Extract the infinitesimal volume element of a sphere of comoving radius r from the Robertson-Walker metric (*2 points*).

d) Calculate the volume of a sphere of comoving radius r in Robertson-Walker geometries with $k = -1, 0, +1$ (*4 points*).

Exercise 4 (10 points)

a) Let us consider a Schwarzshild black hole of mass M and a *static* observer of four-velocity u^μ ($u^i = 0, i = r, \theta, \varphi$). Express u^μ as a function of M and the radial coordinate r (*2 points*).

b) Let us now focus on two points infinitesimally close to the static observer at a). Being infinitesimally close to the static observer, their four-velocity is also u^μ. Let δx^μ be the associated displacement vector. Write down the radial component of the geodesic deviation equation. Express the relevant components of the Riemann tensor in terms of M and r (*6 points*).

c) Calculate the relative radial acceleration between the infinitesimally close points, namely $D^2\delta x^r/D\tau^2$, for $r \to 2MG$. In the limit, $M \to 0$, does $D^2\delta x^r/D\tau^2$ increase or decrease? (*2 points*).