Chapter 0: Introduction and motivation

See .pdf

Chapter I: Terminology

Goal: Introduce / recall some notions / problems.

Def: • A differential equation (DE) is an equation that relates an unknown function and it's derivative.

• An ordinary differential equation (ODE) is a DE, where the unknown function depends on one variable (say $y(x)$, $x(t)$, ..)

• A partial differential equation (PDE) is a DE, where the unknown depends on two or more variables (say $u(x,y,t)$, $y(t,x)$, ...)

Ex (ODE)

Evolution of bacteria is given by a simple model called; Malthusian growth model

$$\frac{d}{dt} P(t) = \lambda \cdot P(t)$$, here λ is a given parameter

$P(t) \to$ amount of bacteria at time t. (unknown)

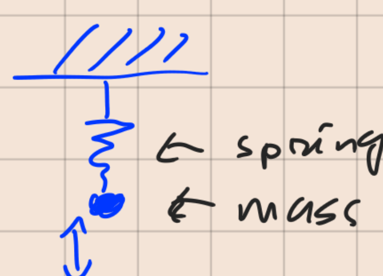
$\dot{P}(t) = \frac{d}{dt} P(t)$ (notation)

Solution? $P(t) = C e^{\lambda t}$, where C is an integrating const.

$$\left[\frac{d}{dt} P(t) = C \cdot \lambda \cdot e^{\lambda t} = \lambda \cdot \underbrace{C \cdot e^{\lambda t}}_{P(t)} = \lambda \cdot P(t) \right]$$

Ex (ODE)

Mass-spring system



$? : x(t)$

displacement of mass by equilibrium

Newton's law: mass × acceleration = Force

$$m \cdot \frac{d^2}{dt^2} x(t) = -k \cdot x(t),$$

m mass (given), k stiffness constant spring (given)

In order to determine a unique sol. to a DE, one needs additional conditions.

Ex (IVP)

Adding an initial value, gives us an initial value problem (IVP).

For bacteria: $\begin{cases} \dot{P}(t) = \lambda P(t) \\ P(0) = P_0 \end{cases}$, DE / IC (= initial condition)

where P_0 (given) is the size of population at initial time ($t=0$)

Sol. $P(t) = P_0 \cdot e^{\lambda t}$.

Ex (BVP)

Adding boundary conditions gives a boundary value problem (BVP). For instance:

$$\begin{cases} -u''(x) = \cos(x) & \text{for } 0 < x < 1 \quad \text{(DE)} \\ u(0) = 0, \ u(1) = 22 & \text{(BC)} \quad \text{Boundary cond.} \end{cases}$$

Next, we define Laplace operator

(2D) $\Delta u(x,y) := \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2}{\partial y^2} u(x,y) = u_{xx}(x,y) + u_{yy}(x,y)$

(3D) $\Delta u(x,y,z) = u_{xx}(x,y,z) + u_{yy}(x,y,z) + u_{zz}(x,y,z)$.

Ex: $u(x,y) = 2xy^3$ (2D)

$$\Delta u(x,y) = u_{xx} + u_{yy} = (2xy^3)_{xx} + (2xy^3)_{yy} =$$

$$= 0 + 2x(6y) = 12xy \,//$$

Ex (PDE)

- Laplace equation: $\Delta u = 0$

  (2D: Find $u(x,y)$ s.t. $\Delta u(x,y) = 0$)

- Heat equation: $u_t - \Delta u = f$

  (1D: $u_t(x,t) - u_{xx}(x,t) = \sin(t)$)

- **Wave equation:** $u_{tt} - \Delta u = g$

  (1D: $u_{tt}(x,t) - u_{xx}(x,t) = 5t$)

Again, in order to determine the exact sol. to a PDE, one needs additional conditions

Ex: (Heat equation on $(0,1)$)

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0 & 0<x<1,\ 0<t<5 & \text{(DE)} \\ u(0,t)=0,\ u(1,t)=7 & 0<t<5 & \text{(BC)} \\ u(x,0) = x^2 & 0<x<1 & \text{(IC)} \end{cases}$$

Classification of second-order linear PDE in 2D:

Consider first the following PDE:

(*) $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, where

$u = u(x,y)$ (unknown), $A,B,C,D,E,F,G \in \mathbb{R}$ (given).

Define discriminant $d = B^2 - 4AC$

One can then classify PDEs of the form (*) by:

Def: The PDE (*) is called

(i) Elliptic if $d < 0$

(ii) Parabolic if $d = 0$

(iii) Hyperbolic if $d > 0$.

Ex:

- Laplace eq. $u_{xx} + u_{yy} = 0 \to A = 1, B = 0, C = 1$

  $\to d = B^2 - 4AC = -4 < 0 \Rightarrow$ elliptic equation

- Heat eq. $u_t - u_{xx} = 0 \xrightarrow[\text{"}x,t \mapsto x,y\text{"}]{u = u(x,t)} A = -1, B = 0, C = 0$

  $\to d = B^2 - 4AC = 0 \Rightarrow$ parabolic equation

- Wave eq. $u_{tt} - u_{xx} = 0 \Rightarrow$ hyperbolic equation.

Rem: The above can be generalised to non-constant $A, B, C, \ldots$, f.ex.

$A = A(x,y)$, $B = B(x,y)$, $C = C(x,y)$, ...

One then defines the type of eq.

at a fixed point (x_0, y_0).

Ex: Triconi equation in gas dynamics

$$y u_{xx} + u_{yy} = 0$$, see book.