Recall: • $(V, (\cdot,\cdot)_V)$ inner product space

• $1 \leq p < \infty$, $\Omega \subset \mathbb{R}^n$ [think $\Omega = [0,1]$, $\Omega = \mathbb{R}^n$ ok!]

$L^p(\Omega) = \{ f : \Omega \to \mathbb{R} : \|f\|_{L^p} < \infty \}$, where

$$\|f\|_{L^p} = \|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p}$$

⚠ $p = 2$ $(p = \infty)$, inner product $(f,g)_{L^2} = \int_\Omega f(x) g(x) \, dx$

Ex: Compute the L^1-norm of $f(x) = 2x + \sin(x)$

for $x \in [0, \frac{\pi}{2}]$.

$$\|f\|_{L^1([0,\frac{\pi}{2}])} \underset{\text{Def}}{=} \int_0^{\pi/2} |f(x)| \, dx \underset{\text{def } f}{=} \int_0^{\pi/2} |\underbrace{2x}_{\geq 0} + \underbrace{\sin(x)}_{\geq 0}| \, dx =$$

$$= \int_0^{\pi/2} (2x + \sin(x)) \, dx = x^2 - \cos(x) \Big|_0^{\pi/2} =$$

$$= \frac{\pi^2}{4} - 0 - 0 + 1 = 1 + \frac{\pi^2}{4}$$

5) Spaces of differentiable functions:

Let $\Omega \subset \mathbb{R}^n$ bounded and open.

We define the following spaces:

Def: • $C(\Omega) = C^0(\Omega) = C^{(0)}(\Omega) = \mathcal{C}(\Omega) =$

$= \{ f : \Omega \to \mathbb{R} : f \text{ continuous} \}$

• $C^1(\Omega) = \{ f : \Omega \to \mathbb{R} : f \text{ is continuously differentiable} \}$

• $C^2(\Omega) = \{ f : \Omega \to \mathbb{R}$: every components of f have all partial derivatives of order ≤ 2 continuous $\}$

• $C^k(\Omega) = \{ f : \Omega \to \mathbb{R} ; D^\alpha f$ are continuous $\forall |\alpha| \leq k \}$

Recall: $D^\alpha f(x) = \dfrac{\partial^{|\alpha|} f(x)}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}}$ where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. $\alpha_j \in \{0, 1, 2, \ldots\}$

• Similarly, we define

$C^k(\bar{\Omega}) = \{ f \in C^k(\Omega) : D^\alpha f$ can be extended from Ω to its closure $\bar{\Omega} \}$

Recall: $\bar{\Omega}$ is the smallest closed set that contains Ω.

- We equip the space $C^k(\bar\Omega)$ with the supremum norm:

$$\|f\|_{C^k(\bar\Omega)} = \sum_{|\alpha| \le k} \sup_{x \in \bar\Omega} |D^\alpha f(x)|$$

Ex: Consider $n=1$, $\Omega = [3,4]$ and $k=1$.

$\Omega = \bar\Omega$ (since Ω closed)

$$\|f\|_{C^1(\bar\Omega)} = \sum_{|\alpha| \le 1} \sup_{x \in [3,4]} |D^\alpha f(x)| =$$

$$= \underbrace{\sup_{x \in [3,4]} |f(x)|}_{(\alpha=0)} + \underbrace{\sup_{x \in [3,4]} |f'(x)|}_{(\alpha=1)}$$

Ex: Consider $n=1$, $\Omega = (1,2)$ open intervall, $f(x) = \frac{1}{x-1}$

$f \in C(\Omega)$? Yes!

$f \in C^1(\Omega)$? Yes! ... $f \in C^k(\Omega)$ $\forall k$

$\bar\Omega = [1,2]$, $f \in C(\bar\Omega)$? No, $f(x) \xrightarrow[x \to 1]{} \infty$

Similarly, $f \notin \mathcal{C}^k(\bar{\Omega})$.

Ex: $n=2$ $\quad f(x) = f(x_1, x_2)$

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$$

Def

$|\alpha| \leq 2 \rightsquigarrow |\alpha| = \alpha_1 + \alpha_2 \leq 2$

$2+0 \rightsquigarrow \frac{\partial^2}{\partial x_1^2}$

$\frac{\partial^2 f}{\partial x_1 \partial x_2} \leftarrow 1+1$

$0+2$

$0+1$

$f(x) \leftarrow 0+0$

$1+0 \rightsquigarrow \frac{\partial}{\partial x_1}$

6) Sobolev spaces:

Let $\Omega \subset \mathbb{R}^n$ open, $k \geq 0$ integer, $1 \leq p \leq \infty$.

Def: • The Sobolev space of order k are defined by

$$W^{k,p}(\Omega) = \{ f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega) \ \forall |\alpha| \leq k \}$$

(weak derivative)

with the norm

$$\|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p}^p \right)^{1/p} \quad \text{where } 1 \leq p < \infty$$

$$\|f\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_{L^\infty(\Omega)}$$

- Later we shall also make use of the semi-norms:

[$|f| = 0$ $\not\Rightarrow$ $f = 0$]

$$|f|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| = k} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p} \qquad 1 \leq p < \infty$$

$$|f|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| = k} \|D^\alpha f\|_{L^\infty(\Omega)} \qquad p = \infty$$

- For $p = 2$, one notes $W^{k,2}(\Omega) = H^k(\Omega)$, i.e.

$$H^k(\Omega) = \{ f \in L^2(\Omega) : D^\alpha f \in L^2(\Omega),\ |\alpha| \leq k \}$$

scalar product $(f, g)_{H^k} = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha f(x) D^\alpha g(x)\,dx$

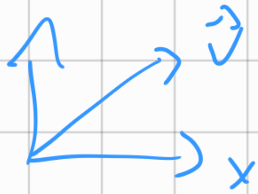
7) Important inequalities:

Recall: For $u, v \in \mathbb{R}^n$, recall

triangle inequality : $\|u + v\| \overset{\Delta}{\leq} \|u\| + \|v\|$

Cauchy-Schwarz : $|(u,v)| \overset{C\text{-}S}{\leq} \|u\| \cdot \|v\|$

Vectors



, $\|\vec v\| \sim$ length

$\downarrow$

$\|\vec v - 0\| \rightsquigarrow$ measure distance 0 $\vec v$

Functions?

$\|f - g\|_{??} \rightsquigarrow$ measure distance

$\|u_{\substack{\text{exact}\\ \text{sol.}}} - u_{\substack{\text{numerical}\\ \text{sol}}}\|_{??} \rightsquigarrow$ measure of error

$L^p, C^k, W^{k,p}$

Def/Theorem: (Minkowski inequality)

Let $1 \le p < \infty$ and $f, g \in L^p(\Omega)$, then one has

$$\|f + g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}$$

Def: A pair of real numbers $p, q \in [1, \infty)$ is called conjugate if $\frac{1}{p} + \frac{1}{q} = 1$

Ex: $p = 2 = q$.

Def/Th: Let p, q be conjugate and $f \in L^p(\Omega)$, $g \in L^q(\Omega)$

then, one has Hölder's inequality

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}$$

$$\int_\Omega |f(x) g(x)| \, dx \leq \left(\int_\Omega |f(x)|^p dx \right)^{1/p} \left(\int_\Omega |g(x)|^q dx \right)^{1/q}$$

($p = 2 = q \rightsquigarrow$ C-S)

Later in the lecture, we shall use the following results:

Th: (Poincaré inequality)

Let $L > 0$ and consider $\Omega = (0, L)$. One has

$$\|u\|_{L^2(\Omega)} \leq C_L \cdot \|u'\|_{L^2(\Omega)} \quad \forall \, u \in H_0^1(\Omega),$$

where $H_0^1(\Omega) = \{ u : \bar{\Omega} \to \mathbb{R} \; : \; u \in H^1(\Omega), \; u(0) = 0 = u(L) \}$

Sobolev space

and

$C_L > 0$ constant (depends L)

Rem: Poincaré tells that we can bound u in terms of its derivatives.

Proof: For $u \in H_0^1 = H_0^1(\Omega)$, one has $u(0) = 0$ (Def)

and then, one has

$$u(x) = \int_0^x u'(\xi)\,d\xi$$ (Fundamental th. of calculus)

Next, compute

$$|u(x)|^2 = \left|\int_0^x 1\cdot u'(\xi)\,d\xi\right|^2 \overset{C\text{-}S}{\leq} \int_0^x 1^2\,d\xi \int_0^x |u'(\xi)|^2\,d\xi \leq$$

$$\underset{x\leq L}{\leq} \int_0^x 1\,d\xi \cdot \int_0^L |u'(\xi)|^2\,d\xi \leq x \cdot \int_0^L |u'(\xi)|^2\,d\xi$$

Finally, we integrate over x!

$$\underbrace{\int_0^L |u(x)|^2\,dx}_{\|u\|_{L^2}^2} \leq \int_0^L \left(x \cdot \int_0^L |u'(\xi)|^2\,d\xi\right)dx$$

$$\leq \int_0^L x\,dx \cdot \int_0^L |u'(\xi)|^2\,d\xi \leq$$

$$\leq \frac{L^2}{2} \cdot \underbrace{\int_0^L |u'(\xi)|^2\,d\xi}_{\|u'\|_{L^2}^2 \text{ (def norm)}}$$

$$\Rightarrow \|u\|_{L^2} \leq \underbrace{\frac{L}{\sqrt{2}}}_{= C_L} \|u'\|_{L^2} \quad :-)$$

$\blacksquare$

Th (Trace theorem)

Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ "nice".

For $u \in W^{1,p}(\Omega)$, one has

$$\|u\|_{L^p(\partial\Omega)} \leq C \cdot \|u\|_{L^p(\Omega)}^{1-1/p} \cdot \|u\|_{W^{1,p}(\Omega)}^{1/p}$$

For $p=2$: $\quad \|u\|^2_{L^2(\partial\Omega)} \leq C \cdot \|u\|_{L^2(\Omega)} \cdot \|u\|_{H^1(\Omega)}$

can estimate u on the boundary $\partial\Omega$ using information on u on domain Ω.

[$\Omega = (0,1) \rightsquigarrow \partial\Omega = \{0,1\}$]

Finally, we provide

Th (Grönwall's inequalities)

Let $\alpha, \beta, u : [0,\infty) \to \mathbb{R}$. Assume α, β, u continuous and

$$u(t) \leq \alpha(t) + \int_0^t \beta(s) u(s)\, ds \qquad \forall t \geq 0$$

a) If β is non-negative, then

$$u(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(r)\, dr\right) ds \quad \forall t$$

b) If in addition α is non-decreasing, then

$$u(t) \leq \alpha(t) \cdot \exp\left(\int_0^t \beta(s)\, ds\right) \quad \forall t \geq 0.$$