Recall: • $L^2(\Omega) = \{ f : \Omega \to \mathbb{R} : \|f\|_{L^2} < \infty \}$,

$$\|f\|_{L^2} = \sqrt{(f,f)_{L^2}} = \left(\int_\Omega |f(x)|^2 \, dx \right)^{1/2}$$

• $H^1(\Omega) = \{ f : \Omega \to \mathbb{R} : \|f\|_{H^1} < \infty \}$,

$$\|f\|_{H^1}^2 = \|f\|_{L^2}^2 + \|f'\|_{L^2}^2$$

• $H_0^1(\Omega) = \{ f \in H^1(\Omega) : f(0) = 0, f(1) = 0 \}$

$$\begin{cases} -u''(x) = f(x) \\ u(0) = 0 \\ u(1) = 0 \end{cases}$$

• Poincaré: $\Omega = (0, L)$

$$\|u\|_{L^2(\Omega)} \leq C_L \cdot \|u'\|_{L^2(\Omega)} \quad \forall u \in H_0^1(\Omega)$$

• C-S:

$$|(f,g)_{L^2}| \leq \|f\|_{L^2} \cdot \|g\|_{L^2}$$

8) Strong form, weak and variational form, and minimisation:

Consider Poisson's equation

$$(BVP)\quad \begin{cases} -u''(x) = f(x) & \text{for } x \in \Omega := (0,1) \\ u(0) = 0,\ u(1) = 0 \end{cases}$$

Here f is given (continuous + bounded).

We look for $u \in C^2(\Omega)$.

The above formulation of the problem is called the strong form of the problem.

Rem: Book $\quad -\left(a(x)u'(x)\right)' = f(x)$

Rem: The finite element method, see later, is based on a reformulation of (BVP).

We next present a reformulation of (BVP).

Idea (Galerkin)

Consider $H_0^1(\Omega) = \{ v \in H^1(\Omega) : v(0) = 0,\ v(1) = 0 \}$

We then multiply (BVP) with a test function $v \in H_0^1$ and integrate over $\Omega = (0,1)$:

$$-\int_0^1 u''(x)\, v(x)\, dx = \int_0^1 f(x)\, v(x)\, dx \qquad \forall v \in H_0^1$$

Do an integration by part:

$$\underbrace{-u'(x)\, v(x) \Big|_0^1}_{-u'(1)\underset{0}{\underset{\shortparallel}{v(1)}} + u'(0)\underset{0}{\underset{\shortparallel}{v(0)}} = 0 \text{ since } v \in H_0^1} + \int_0^1 u'(x)\, v'(x)\, dx = \int_0^1 f(x)\, v(x)\, dx \quad \forall v \in H_0^1$$

Finally, we obtain the weak or variational form of (BVP):

(VF) Find $u \in H_0^1$ s.t. $\displaystyle\int_0^1 u'(x)\, v'(x)\, dx = \int_0^1 f(x)\, v(x)\, dx \quad \forall v \in H_0^1$

compact notation: $(u', v')_{L^2} = (f, v)_{L^2} \quad \forall v \in H_0^1$

(def inner product L^2)

But there is another way to look at the problem:

Idea (Ritz)

Consider $F(v) = \frac{1}{2}(v', v')_{L^2} - (f, v) \qquad \forall v \in H_0^1$

Why?

Let u be a solution (VF) and compute

linearity

$$F(u+v) = \frac{1}{2}\left(u'+v', u'+v'\right)_{L^2} - (f, u+v)_{L^2} =$$

$$= \underline{\frac{1}{2}(u',u')_{L^2}} + \frac{1}{2}(v',v')_{L^2} + \underline{\frac{2}{2}(u',v')_{L^2}} - \underline{(f,u)} - \underline{(f,v)}$$

$$= \underbrace{(u',v')_{L^2} - (f,v)_{L^2}}_{\overset{\shortparallel}{0} \text{ since } u \text{ sol. (VF)}} + \underbrace{\frac{1}{2}(u',u')_{L^2} - (f,u)}_{F(u)} + \underbrace{\frac{1}{2}(v',v')}_{\geq 0}$$

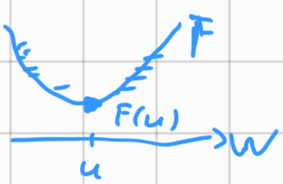
$$\geq F(u)$$

Thus $F(\underbrace{u+v}_{\in H_0^1}) \geq F(u) \quad \forall v \in H_0^1$

$F : H_0^1 \to \mathbb{R}$

$\Rightarrow F(w) \geq F(u) \;\; \forall w \in H_0^1$

This gives us a minimisation problem:



(MP) Find $u \in H_0^1$ s.t. $F(u)$ is minimised.

$F(w) = \frac{1}{2}(w', w') - (f, w)$

<u>Summary:</u> "u sol. (BVP)/strong" $\Rightarrow$ "u sol. (VF)/weak"

$\Rightarrow$ "u sol. to (MP)"

$? \Leftarrow ?$

<u>(MP) $\Rightarrow$ (VF):</u>

We start with u sol. to (MP): $F(u) \leq F(\underbrace{u+\varepsilon v}_{\in H_0^1})$ $\forall \varepsilon \in \mathbb{R}$ $\forall v \in H_0^1$

Idea (Euler) Consider $F(u+\varepsilon v)$ as a function of ε!

It means if u minimise $F(u)$, $\varepsilon = 0$ is a minimum for $F(u+\varepsilon v)$. Hence, $\frac{\partial}{\partial \varepsilon} F(u+\varepsilon v)\Big|_{\varepsilon=0} = 0$.

We compute $F(u+\varepsilon v) \underset{\text{Def } F}{=} \frac{1}{2}(u'+\varepsilon v', u'+\varepsilon v')_{L^2} - (f, u+\varepsilon v)_{L^2} =$

$$= \frac{1}{2}(u',u')_{L^2} + \frac{\varepsilon^2}{2}(v',v')_{L^2} + \frac{2\varepsilon}{2}(u',v')_{L^2} - (f,u)_{L^2} - \varepsilon(f,v)_{L^2}$$

We derive:

$$\frac{\partial}{\partial \varepsilon} F(u+\varepsilon v)\Big|_{\varepsilon=0} = \varepsilon(v',v')_{L^2} + (u',v')_{L^2} - (f,v)_{L^2}\Big|_{\varepsilon=0} =$$

$$= (u',v')_{L^2} - (f,v)_{L^2} \overset{!}{=} 0$$

Thus $(u',v')_{L^2} = (f,v)_{L^2}$ $\forall v \in H_0^1$

This means that u is also sol. to (VF).

(VF) $\Rightarrow$ (BVP): assuming in addition that $u \in C^2(\Omega)$.

From (VF), we know that $\underbrace{(u',v')_{L^2}}_{\int u'v'} = (f,v)_{L^2}$ $\forall v \in H_0^1$

Doing an integration by part gives us:

$$\underbrace{-u'(x)v(x)\Big|_0^1}_{=0 \text{ since } v\in H_0^1} - \int_0^1 u''(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx \quad \forall v\in H_0^1$$

This gives us : $-\int_0^1 u''(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx \quad \forall v\in H_0^1$

or $\int_0^1 (u''(x)+f(x))\,v(x)\,dx = 0 \quad \forall v\in H_0^1 \; (*)$

Now we show that, by contradiction, $(*)$ implies $u''(x)+f(x)=0$ for $x\in\Omega=(0,1)$, i.e. (BVP).

Assume, by contradiction, that $u''+f\neq 0$. Since $u\in C^2(\Omega)$,

$u''+f$ is continuous and then $\exists\, x\in\Omega$ s.t.

$u''+f\neq 0$ in a neighborhood of this x.

We define $v\in H_0^1$ s.t. it has the same sign as $u''+f$

on this neighborhood and zero else. Now we look

$$\int_0^1 \underbrace{(u''(x)+f(x))\,v(x)}_{>0 \text{ on the neighborhood}}\,dx \neq 0$$

This contradicts $(*)$ and

thus we must have $u''(x)+f(x)=0$ for $x\in\Omega$, i.e. u is sol. to (BVP) / strong.

$\Rightarrow$ Th! Strong/(BVP) $\Longrightarrow$ weak/(VF) $\Longleftrightarrow$ (MP)
$\Longleftarrow$ $\oplus$ $u \in C^2(\Omega)$

But ... does (VF) have a unique sol. ??

9) Lax-Milgram theorem:

Recall: Poisson's equation: (BVP) $\begin{cases} -u''(x) = f(x) & x \in (0,1) = \Omega \\ u(0) = 0, u(1) = 0 \end{cases}$

(VF) Find $u \in H_0^1(\Omega)$ s.t. $\underbrace{(u', v')_{L^2}}_{=: a(u,v)} = \underbrace{(f, v)_{L^2}}_{\ell(v)}$, $\forall v \in H_0^1(\Omega)$

where $a(u,v) := (u', v')_{L^2}$, $\ell(v) = (f, v)_{L^2}$

We can generalise this kind of problem, by looking at the following:

$$\boxed{\begin{array}{l} \text{Find } u \in H \text{ s.t. } a(u,v) = \ell(v) \quad \forall v \in H, \\ \text{where } \quad H \text{ is a Hilbert space} \\ \qquad a: H \times H \to \mathbb{R} \text{ is a bilinear form, bounded, coercive} \\ \qquad \ell: H \to \mathbb{R} \text{ is a bounded linear functional} \end{array}}$$

Def: • A Hilbert space is an inner product space that is complete [i.e. all Cauchy sequence converge in this space]

Ex: $L^2(\Omega)$, $H^1(\Omega)$, $H^1_0(\Omega)$ are Hilbert space.

• $a: H \times H \to \mathbb{R}$ is bilinear if $a(\lambda u + \mu v, w) = \lambda a(u,w) + \mu a(v,w)$
and $a(u, \lambda v + \mu w) = \lambda a(u,v) + \mu a(u,w)$
$\forall \lambda, \mu \in \mathbb{R}$, $\forall u,v,w \in H$.

• $a: H \times H \to \mathbb{R}$ is bounded if $\exists \alpha > 0$ s.t.

$$a(u,v) \leq \alpha \cdot \|u\|_H \cdot \|v\|_H \quad \forall u,v \in H$$

• $a: H \times H \to \mathbb{R}$ is coercive if $\exists \beta > 0$ s.t.

$$a(u,u) \geq \beta \|u\|_H^2 \quad \forall u \in H$$

• $\ell: H \to \mathbb{R}$ is bounded if $\exists \gamma > 0$ s.t.

$$|\ell(v)| \leq \gamma \cdot \|v\|_H \quad \forall v \in H$$

Rem: $a(\cdot,\cdot)$ does not need to be symmetric!

Th (Lax-Milgram)
Let a, ℓ be as above. Then there exists a unique element $u \in H$ s.t. $a(u,v) = \ell(v)$ $\forall v \in H$.

We shall use (LM) to show that (WF) to Poisson's eq. has a unique solution!!