Recall: • C-S $|(f,g)_{L^2}| \leq \|f\|_{L^2} \cdot \|g\|_{L^2}$

• $\|f\|_{H^1}^2 = \|f\|_{L^2}^2 + \|f'\|_{L^2}^2$

• Poincaré : $\|u\|_{L^2(0,1)} \leq \frac{1}{\sqrt{2}} \|u'\|_{L^2(0,1)} \quad \forall u \in H_0^1(0,1)$

• Poisson's equation

(BVP) strong $\begin{cases} -u''(x) = f(x) & \text{for } x \in (0,1) \\ u(0)=0,\ u(1)=0 \end{cases}$

Weak/VF Find $u \in H_0^1$ s.t. $(u', v')_{L^2} = (f, v)_{L^2} \quad \forall v \in H_0^1$

MP Find $u \in H_0^1$ s.t. $F(v) := \frac{1}{2}(v', v')_{L^2} - (f, v)_{L^2}$ is min. in H_0^1

• Lax - Milgram Th.

$a(\cdot,\cdot)$ bounded, coercive, bilinear form
$a(u,v) \leq \alpha \|u\|_H \cdot \|v\|_H$, $a(u,u) \geq \beta \|u\|_H^2$

$\ell(\cdot)$ bounded linear functional
$|\ell(v)| \leq \gamma \cdot \|v\|_H$

H Hilbert space

$\Rightarrow \exists!\ u \in H$ s.t. $a(u,v) = \ell(v) \quad \forall v \in H$

We now use (LM) in order to show that the (VF) of Poisson's equation has a unique solution

<u>Th</u>: Let $f \in L^2(0,1)$. Then there exists a unique sol. to the (VF) of Poisson's eq:

Find $u \in H_0^1(0,1)$ s.t. $(u', v')_{L^2} = (f, v)_{L^2}$ $\forall v \in H_0^1$

<u>Proof</u>:

- Set $a(u,v) := (u', v')_{L^2}$ and $\ell(v) := (f, v)_{L^2}$ for $u, v \in H_0^1$.

- Let us check that $a(\cdot,\cdot)$ is bounded:

$$a(u,v) \underset{\text{Def of } a(\cdot,\cdot)}{=} (u', v')_{L^2} \underset{C\text{-}S}{\leq} \|u'\|_{L^2} \cdot \|v'\|_{L^2} \underset{\text{Def } H^1\text{-norm}}{\leq} \|u\|_{H^1} \cdot \|v\|_{H^1}$$

$\rightarrow$ $a(\cdot,\cdot)$ is bounded (with $\alpha = 1$).

- Next, we show that $a(\cdot,\cdot)$ is coercive.

Recall that Poincaré tells us $\|u\|_{L^2} \leq \frac{1}{\sqrt{2}} \|u'\|_{L^2}$ $\forall u \in H_0^1$

Consider the following

$$\|u\|_{H^1}^2 \underset{\uparrow}{=} \|u\|_{L^2}^2 + \|u'\|_{L^2}^2 \underset{\uparrow}{\leq} \frac{1}{2}\|u'\|_{L^2}^2 + \|u'\|_{L^2}^2 \leq$$

Def H^1-norm Poincaré

$$\leq \frac{3}{2}\|u'\|_{L^2}^2 \underset{\uparrow}{=} \frac{3}{2}(u',u')_{L^2} \overset{\downarrow}{=} \frac{3}{2}a(u,u)$$

Def $a(\cdot,\cdot)$

Def L^2-norm $\|v\|_{L^2} = \sqrt{(v,v)_{L^2}}$

This shows that $a(\cdot,\cdot)$ is coercive $(\beta = \frac{3}{2})$.

- Show that $\ell(\cdot)$ is bounded:

$$|\ell(v)| \underset{\uparrow}{=} |(f,v)_{L^2}| \underset{\uparrow}{\leq} \underbrace{\|f\|_{L^2}}_{\leq \gamma < \infty} \cdot \|v\|_{L^2} \leq \gamma \cdot \|v\|_{L^2}$$

Def of ℓ C-S $\leq \gamma < \infty$ $(f \in L^2)$

$L^2 = \{f : [0,1] \to \mathbb{R} : \|f\|_{L^2} < \infty\}$

$$\underset{\uparrow}{\leq} \gamma \cdot \|v\|_{H^1}$$

Def of H^1-norm

This shows that $\ell(\cdot)$ is bounded

$\xrightarrow{LM}$ $\exists!\ u \in H_0^1$ s.t. $a(u,v) = \ell(v)$ $\forall v \in H_0^1$

$(u',v')_{L^2} = (f,v)_{L^2}$

Rem: ⚠ $a(\cdot,\cdot)$ is coercive in H_0^1 not H^1 !!

Because $a(u,u) = (u',u')_{L^2}$ is zero for

constant function $u \equiv const \neq 0$ without $\|u\|_{H^1} = 0$.

Chapter III: Numerical methods for IVP

Goal: Present a few basic numerical methods for

Generalisation : $\dot{y}(t) = f(t, y(t))$

$$(IVP) \quad \begin{cases} \dot{y}(t) = f(y(t)) & \text{for } 0 < t \leq T \\ y(0) = y_0 \end{cases}$$

y_0, f, T are given, unknown $y = ??$

Euler: $y_1 = y_0 + k \cdot f(t_0, y_0)$

1) Motivation: Finite difference schemes

Consider $y: \mathbb{R} \to \mathbb{R}$ differentiable and $t_0 \in \mathbb{R}$.

By definition,

$$\dot{y}(t_0) = \lim_{h \to 0} \frac{y(t_0+h) - y(t_0)}{h}$$

For a fixed (small) $h > 0$, the terme $\dfrac{y(t_0+h) - y(t_0)}{h}$

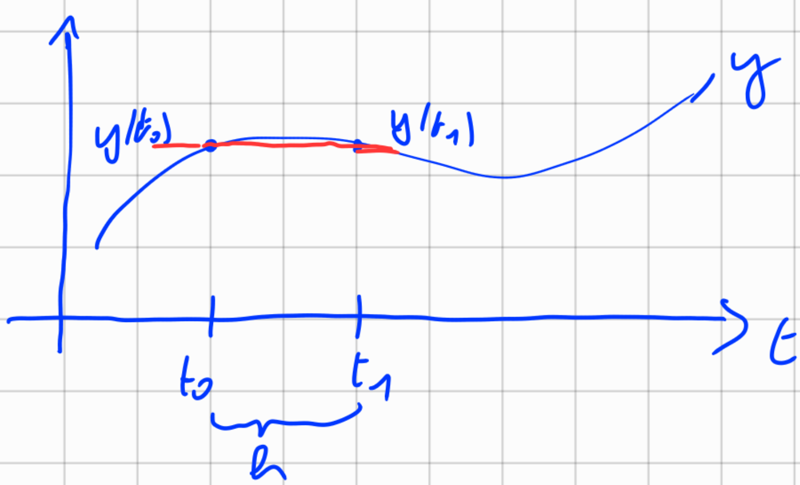
will be a good approximation of $\dot{y}(t_0)$.

Def: $\dot{y}(t_0) \approx \frac{y(t_0+h)-y(t_0)}{h}$ which named a <u>forward difference</u>.

approximation

Let us define $t_1 := t_0 + h$ $\Rightarrow$ above reads

$$\dot{y}(t_0) \approx \underbrace{\frac{y(t_1)-y(t_0)}{t_1-t_0}}_{\text{slope of the line connecting } y(t_1) \text{ and } y(t_0)}$$



Similarly, one defines

<u>Backward difference</u> $\dot{y}(t_0) \approx \frac{y(t_0)-y(t_0-h)}{h}$

<u>Centered difference</u> $\dot{y}(t_0) \approx \frac{y(t_0+h)-y(t_0-h)}{2h}$

2) First numerical schemes for IVP:

Idea (Euler 1768)

Consider $N \in \mathbb{N}$ (BIG), define time step $k = \frac{T}{N}$ (small)

and a grid in time $0 = t_0 < t_1 < t_2 < \dots < t_N = T$, where

$t_{n+1} - t_n = k$.

For $t_0 \leq t \leq t_1 = t_0 + k$, approximate $\dot{y}(t)$ by

forward difference: $\underset{\substack{\parallel \\ f(y(t))}}{\dot{y}(t)} \approx \frac{y(t+k) - y(t)}{k}$

by def. of (IVP).

This gives us the approximation

$$y(t+k) \approx y(t) + k \cdot f(y(t))$$

Take $t = t_0$ above:

$$y(t_0 + k) \approx y(t_0) + k \cdot f(y(t_0))$$

or

$$\underset{\substack{\parallel \\ \text{exact sol.} \\ ??}}{y(t_1)} \approx \underbrace{y_0 + k \cdot f(y_0)}_{\substack{\text{numerical sol.} \\ \text{can compute this!!}}} =: y_1$$

$\hookrightarrow \quad y(t_1) \approx y_1 = y_0 + k \cdot f(y_0)$

We can repeat this procedure and obtain

Explicit / forward Euler scheme:

$$t_{n+1} = t_n + k$$

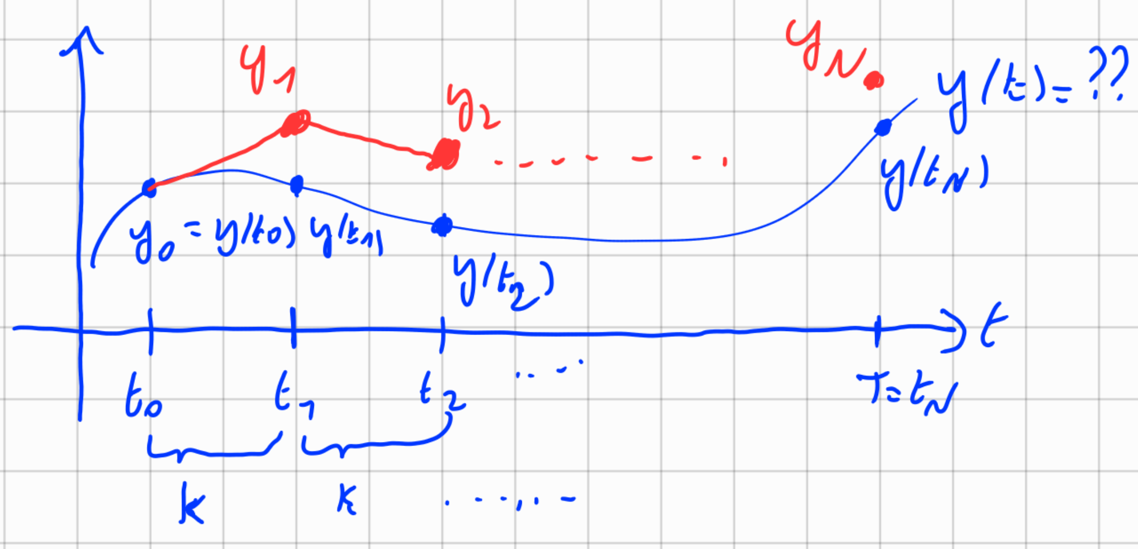
$$y_{n+1} = y_n + k \cdot f(y_n) \qquad \text{for } n = 0, 1, \dots, N-1$$

$$\left[y_{n+1} = y_n + k\, f(t_n, y_n) \right]$$

which provides numerical approximation

$$y_n \approx y(t_n) \quad \text{on grid } t_n \text{ for } n = 0, 1, \dots, N$$

numerical $\qquad$ exact (??)



Similarly, one defines

Backward/implicit Euler $\qquad y_{n+1} = y_n + k\cdot f(y_{n+1}) \qquad n=0,1,2,\dots,N$

Solve nonlinear system

Crank-Nicolson scheme

$$y_{n+1} = y_n + \underbrace{\frac{k}{2}\left(f(y_n) + f(y_{n+1})\right)}_{\text{average of } f}$$

Chapter IV: A Galerkin FEM for BVP

Goal: Introduce finite element methods (FEM)

1) Space of continuous piecewise linear functions

Consider a partition of the interval $[0,1]$ into

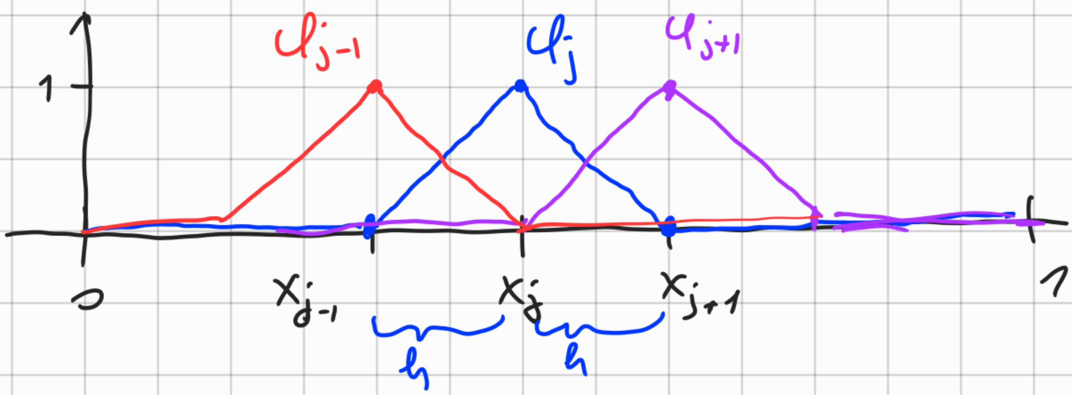
$m+1$ subintervals (x_{j-1}, x_j):

T_h: $0 = x_0 < x_1 < x_2 < \dots < x_m < x_{m+1} = 1$, where

$h_j = x_j - x_{j-1}$ for $j = 1, 2, \dots, m+1$.

Next, define hat functions $\{\varphi_j\}_{j=0}^{m+1}$ defined by

$$\varphi_j(x) = \begin{cases} \dfrac{x-x_{j-1}}{h_j} & \text{for } x_{j-1} \le x \le x_j \\ \dfrac{x-x_{j+1}}{-h_{j+1}} & \text{for } x_j \le x \le x_{j+1} \\ 0 & \text{else} \end{cases} \qquad \text{for } j = 1, 2, \dots, m$$



Rem: • " $\varphi_j(x_i) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases}$ "

• φ_0 and φ_{m+1} are half hat functions

Def: We define the space of continuous piecewise linear functions on $[0,1]$ by

$$V_h(0,1) = \operatorname{span}(\varphi_0, \varphi_1, \varphi_2, \dots, \varphi_{m+1})$$

We note that every $v \in V_h(0,1)$ can be written as
$$v(x) = \sum_{j=0}^{m+1} \zeta_j \varphi_j(x)$$

coefficients $\zeta_j = v(x_j)$

hat funct. (Basis)

$$\left[\mathbb{R}^n: \quad u = \sum_{j=1}^{n} \zeta_j e_j \right]$$

$\rightsquigarrow$ standard basis