Assignment 1

January 26, 2021

Instructions: You may work in a group of 2 persons but hand in only one report for the group (with both names and relevant information). If you are alone and want to work with someone send an email to David before Tuesday 26.01.21. Submit a concise report (max 10 pages) by February 15 on canvas (submitted report with zero points may be resubmitted one time). Each task below gives one point. To pass the assignment, one needs at least two points. The possibly obtained extra point(s) (at most two points) will be counted as bonus point(s) for the written examination.

The assignment is based on a previous version from 2020 *by M. Asadzadeh.*

1. Write a program that computes the $cG(1)$ finite element approximations of the two-point boundary value problem

$$\begin{cases} -u''(x) = f(x) \quad \text{in} \quad (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where f is a given function by the user. Make sure that the code is as efficient as possible using your knowledge from linear algebra and materials from chapters 3 and 5 of the book. Test your codes for $f(x) = 6x$, where one can find the exact solution to the above BVP.

2. Do the above exercise with a $cG(2)$ finite element approximation.

3. Let $\varepsilon > 0$ and consider the continuous Galerkin $cG(1)$ method for the one-dimensional boundary value problem

$$\begin{cases} -\varepsilon u''(x) + u'(x) = 0 \quad \text{in} \quad (0,1), \\ u(0) = 1, u(1) = 0. \end{cases}$$

(a) Write down the FE problem for a $cG(1)$ numerical approximation computed on a uniform mesh with M interior nodes.

(b) Compute the numerical approximations for $\varepsilon = 0.01$ and with $M = 10$ as well as $M = 11$. Compare with the analytical (exact) solution to the above BVP.

(c) Compute the FE solution when $M = 100$ and compare your result with the analytical solution.

4. Let $T, u_0 \in \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ and consider the initial value problem

$$\begin{cases} \dot{u}(t) + au(t) = f(t) \quad \text{for} \quad 0 < t \le T, \\ u(0) = u_0. \end{cases}$$

(a) Find the exact solution to the above problem for $u_0 = 1, a = 4$ and $f(t) = t^2$.

(b) Compute the $cG(1)$ numerical solution to the above problem and compare to the exact solution (feel free to play a little bit with the parameters).

(c) Consider the above problem for $u_0 = 1, f = 0$, time interval $[0,1]$, $a = 10$ and $a = 100$. Implement explicit Euler, implicit Euler, and Crank–Nicolson's schemes with time step k. For which values of k is explicit Euler stable? Plot your numerical solutions with different time steps, for instance $k = 0.1$, and compare your results.