Solutions to exercises
MMG800

Malin Nilsson

January 27, 2021

Week 1

53. Let $B = \{x \in \mathbb{R}^2 : |x| < 1\}$ be the unit disc in $\mathbb{R}^2$. Find condition on α for which $|x|^\alpha \in H^1(B)$ but $|x|^s \notin H^1(B)$ for any $s < \alpha$.

    Solution. We have that $u = |x|^\alpha \in H^1(B)$ if

$$\underbrace{\int_B |x|^{2\alpha}}_{(1)} + \underbrace{\sum_{i=1}^{2} \int_B \left(\frac{\partial u}{\partial x_i}\right)^2 \mathrm{d}x_1 \mathrm{d}x_2}_{(2)} < \infty.$$

With polar coordinates, we have

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta \text{ and} \\ \mathrm{d}x &= r\mathrm{d}r\mathrm{d}\theta, \end{aligned}$$

giving us

$$\begin{aligned} &(1) = 2\pi \int_0^1 r^{2\alpha} \cdot r\mathrm{d}r = 2\pi \int_0^1 r^{2\alpha+1}\mathrm{d}r < \infty \text{ if} \\ &2\alpha + 1 > -1 \\ &2\alpha > -2 \\ &\alpha > -1. \end{aligned}$$

Further, we have that $(2) < \infty$ if $\int_B \left(\frac{\partial u}{\partial x_1}\right)^2 < \infty$ due to symmetry reasons.

$$\int_B \left(\frac{\partial u}{\partial x_1}\right)^2 \mathrm{d}x_1\mathrm{d}x_2 = \left\{\begin{array}{l} \text{Assume } \alpha \neq 0. \\ \frac{\partial}{\partial x_1}|x|^\alpha = \\ \frac{\partial}{\partial x_1}(x_1^2+x_2^2)^{\alpha/2} = \\ \frac{\alpha}{2}(x_1^2+x_2^2)^{\alpha/2-1}\cdot 2x_1 = \\ \frac{\alpha}{2}r^{\alpha-2}\cdot 2r\cos\theta = \\ \alpha r^{\alpha-1}\cdot\cos\theta \end{array}\right\} =$$

$$= \underbrace{\int_0^{2\pi} \cos^2\theta \mathrm{d}\theta \cdot \alpha \cdot}_{<\infty} \underbrace{\int_0^1 \left(r^{\alpha-1}\right)^2 \cdot r\mathrm{d}r}_{\substack{=\int_0^1 r^{2\alpha-1}\mathrm{d}r<\infty \text{ if} \\ 2\alpha-1>-1 \\ 2\alpha>0 \\ \alpha>0}}$$

And likewise (2) diverges if $\alpha < 0$. If $\alpha = 0$,

$$u = |x|^0 = 1, \qquad \frac{\partial}{\partial x_1}(1) = 0, \qquad \int_B 0\mathrm{d}x = 0.$$

We obtain that $|x|^\alpha \in H^1(B)$ if $\alpha \geq 0$ and $|x|^\alpha \notin H^1(B)$ if $\alpha < 0$.

54. Define $H^2(B)$ and find a function that is in $H^1(B)$ but not in $H^2(B)$, where B is the unit disc.

Solution. The definition of $H^2(B)$ is

$$H^2(B) = \left\{u \in L_2(B) : \frac{\partial u}{\partial x_i} \in L_2(B), \frac{\partial^2 u}{\partial x_i \partial x_j} \in L_2(B), \qquad i,j = 1,2\right\}.$$

Inspired by the last exercise, we choose $u = |x|^{\frac{1}{2}} = (x_1^2 + x_2^2)^{1/4}$. We then have

$$\begin{aligned} \frac{\partial u}{\partial x_1} &= \frac{x_1}{2}(x_1^2+x_2^2)^{-3/4} \\ \frac{\partial^2 u}{\partial x_1^2} &= \frac{1}{2}(x_1^2+x_2^2)^{-3/4} - \frac{3}{4}x_1^2(x_1^2+x_2^2)^{-7/4} = \\ &= \frac{2(x_1^2+x_2^2)-3x_1^2}{4(x_1^2+x_2^2)^{7/4}} \end{aligned}$$

We then have

$$\int\limits_B \left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 \mathrm{d}x = \int\limits_B \frac{(2(x_1^2+x_2^2)-3x_1^2)^2}{16(x_1^2+x_2^2)^{7/2}}\mathrm{d}x = \int\limits_0^{2\pi}\int\limits_0^1 \frac{(2r^2-3r^2\cos^2\theta)^2}{16r^7} r\mathrm{d}r\mathrm{d}\theta =$$
$$= \int\limits_0^{2\pi}\int\limits_0^1 \frac{r^5(2-3\cos^2\theta)^2}{16r^7}\mathrm{d}r\mathrm{d}\theta = \underbrace{\int\limits_0^1 \frac{1}{r^2}\mathrm{d}r}_{\left[\frac{-1}{r}\right]_0^1 \text{ diverges}} \cdot \underbrace{\int\limits_0^{2\pi} \frac{(2-3\cos^2\theta)^2}{16}\mathrm{d}\theta}_{<\infty},$$

so u is not in $H^2(B)$. According to the last exercise, $\int_B \left(\frac{\partial u}{\partial x_1}\right)^2 \mathrm{d}x$ converges. To see it again:

$$\int\limits_B \left(\frac{\partial u}{\partial x_1}\right)^2 \mathrm{d}x = \frac{1}{2}\int\limits_B \left(\frac{x_1}{(x_1^2+x_2^2)^{3/4}}\right)^2 \mathrm{d}x = \frac{1}{2}\int\limits_B \frac{x_1^2}{(x_1^2+x_2^2)^{3/2}}\mathrm{d}x =$$
$$= \frac{1}{2}\int\limits_0^{2\pi}\int\limits_0^1 \frac{r^2\cos^2\theta}{r^3} r\mathrm{d}r\mathrm{d}\theta = \frac{1}{2}\int\limits_0^{2\pi}\int\limits_0^1 \cos^2\theta\mathrm{d}r\mathrm{d}\theta < \infty.$$

55. Prove the Poincaré-Friedrich inequality:

$$\|v\|_{L_2(\Omega)}^2 \le C_\Omega \left(\|v\|_{L_2(\Gamma)}^2 + \|\nabla v\|_{L_2(\Omega)}^2\right), \qquad \forall v \in H^1(\Omega),$$

where $\Omega \subset \mathbb{R}^n$ is bounded with smooth boundary Γ.

Solution. Take $\varphi \in C^2(\mathbb{R}^n)$ s.t. $\Delta\varphi = 1$, e.g. $\varphi = \frac{1}{4}(x^2+y^2)$ in $\mathbb{R}^2$ or $\varphi = \frac{1}{2n}|x|^2$ in $\mathbb{R}^n$. Then

$$\|u\|_{L_2(\Omega)}^2 = \int\limits_\Omega u^2 \Delta\varphi \mathrm{d}x = \left\{ \begin{array}{l} \text{Green's formula:} \\ \int\limits_\Omega (\Delta u)v = \int\limits_{\partial\Omega} (\nabla u \cdot n)v - \int\limits_\Omega \nabla u \cdot \nabla v \end{array} \right\} =$$
$$= \int\limits_\Gamma u^2 \nabla\varphi \cdot n - \int\limits_\Omega 2u\nabla u \cdot \nabla\varphi \le \left\{ \begin{array}{l} \varphi \in C^2(\mathbb{R}^n), \\ \Omega \text{ bounded} \end{array} \right\} \le$$
$$\le c\|u\|_{L_2(\Gamma)}^2 + 2c\,\|u\,|\nabla u|\|_{L_1(\Omega)} =$$
$$= c\|u\|_{L_2(\Gamma)}^2 + 2c\left\|\sqrt{\varepsilon}u \cdot \frac{1}{\sqrt{\varepsilon}}|\nabla u|\right\|_{L_1(\Omega)} \le$$

$$\leq \left\{\ 2ab \leq a^2 + b^2\ \right\} \leq c\,\|u\|^2_{L_2(\Gamma)} + c\left\|\varepsilon u^2 + \frac{1}{\varepsilon}\,|\nabla u|^2\right\|_{L_1(\Omega)} \leq$$
$$\leq \left\{\ \int a^2 + b^2 \mathrm{d}x = \int a^2 \mathrm{d}x + \int b^2 \mathrm{d}x\ \right\} \leq$$
$$\leq c\,\|u\|^2_{L_2(\Gamma)} + \underbrace{c\varepsilon\,\|u\|^2_{L_2(\Omega)}}_{*} + \frac{c}{\varepsilon}\,\|\nabla u\|^2_{L_2(\Omega)}\,.$$

Now, move (*) to the other side and choose $\varepsilon = \frac{1}{2c}$. We get

$$\|u\|^2_{L_2(\Omega)} \leq C\left(\|u\|^2_{L^2(\Gamma)} + \|\nabla u\|^2_{L_2(\Omega)}\right).$$

56. Verify the trace theorem: If Ω is a bounded domain with boundary Γ, then there is a constant C such that

$$\|v\|_{L_2(\Gamma)} \leq C\,\|v\|_{H^1(\Omega)}\,, \qquad \forall v \in H^1(\Omega).$$

Solution. Choose $\varphi \in C^2(\mathbb{R})$ s.t. $n \cdot \nabla\varphi = 1$ on $\partial\Omega$.

$$\int_\Omega u^2 \Delta\varphi = \left\{ \begin{array}{l} \text{Green's formula:} \\ \int_\Omega (\Delta u)v = \int_{\partial\Omega} (\nabla u \cdot n)v - \int_\Omega \nabla u \cdot \nabla v \end{array} \right\} =$$
$$= \int_{\partial\Omega} u^2\ \underbrace{\nabla\varphi \cdot n}_{=1} - \int_\Omega (2u\nabla u)\cdot\nabla\varphi \rightarrow$$
$$\int_{\partial\Omega} u^2 \leq \int_\Omega u^2\,|\Delta\varphi| + \int_\Omega 2\,|u|\,|\nabla u|\,|\nabla\varphi| \rightarrow \left\{ \begin{array}{c} \varphi \in C^2(\mathbb{R}), \\ \Omega \text{ bounded} \end{array} \right\}$$
$$\int_{\partial\Omega} u^2 \leq c\int_\Omega u^2 + 2c\int_\Omega |u|\,|\nabla u| \rightarrow$$
$$\left\{\ 2ab \leq a^2 + b^2\ \right\} \rightarrow$$
$$\|u\|^2_{L_2(\partial\Omega)} \leq C\left(\|u\|^2_{L_2(\Omega)} + \|\nabla u\|^2_{L_2(\Omega)}\right) = C\,\|u\|^2_{H^1(\Omega)}\,.$$

57. Show that there is no constant C s.t. $\|v\|_{L_2(\Gamma)} \leq C\,\|v\|_{L_2(\Omega)}\,, \forall v \in L_2(\Omega)$. **Solution.** We have to construct a counter example. Let Ω be the unit disc B in $\mathbb{R}^2$. We want to construct a series of functions that grow on the border but where the area under the functions converge to some number. We picture the function in figure 1.

Consider the function

$$f_\varepsilon = \frac{|x|}{(1 - |x| + \varepsilon)^{1/4}}.$$

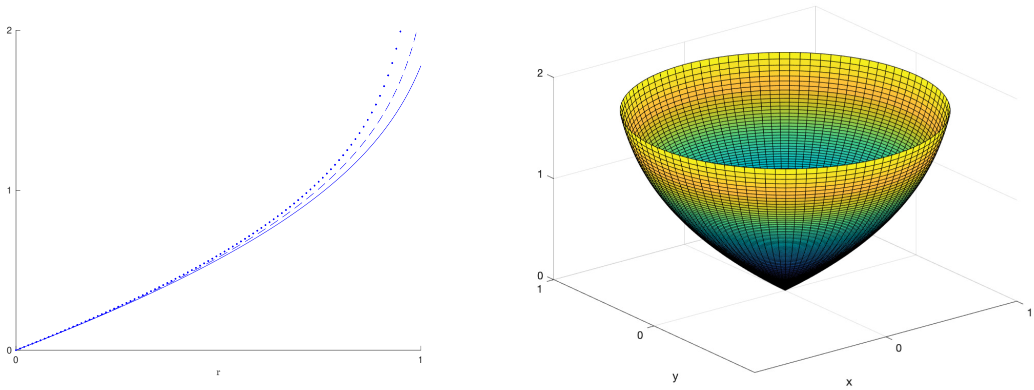

Figure 1: Picture this function.

We then have

$$f_\varepsilon\Big|_{\partial\Omega} = \frac{1}{\varepsilon^{1/4}} \qquad \Rightarrow \qquad \int\limits_{\partial\Omega} f_\varepsilon^2 = \frac{2\pi}{\varepsilon^{1/2}} \to \infty, \varepsilon \to 0^+,$$

but

$$\int\limits_\Omega f_\varepsilon^2 = \{\text{ pol. coord. }\} = \int\limits_0^{2\pi}\int\limits_0^1 \left(\frac{r}{(1-r+\varepsilon)^{1/4}}\right)^2 r\mathrm{d}r\mathrm{d}\theta =$$

$$= 2\pi \int\limits_0^1 \frac{r^3}{(1-r+\varepsilon)^{1/2}}\mathrm{d}r = \ldots =$$

$$= \frac{4\pi}{35}\left(16\underbrace{(\varepsilon+1)^{7/2}}_{\to 1} - \underbrace{\sqrt{\varepsilon}(16\varepsilon^3+56\varepsilon^2+70\varepsilon+35)}_{\to 0}\right) \longrightarrow \frac{\pi}{140}, \varepsilon \longrightarrow 0^+.$$

58. Consider the convection-diffusion problem of finding $u \in H_0^1(\Omega)$ s.t.

$$\begin{aligned} -\nabla\cdot(\varepsilon\nabla u + \beta u) &= f, \qquad && \Omega \subset \mathbb{R}^2 \text{ bounded, convex, polygonal,} \\ u &= 0, && \text{on } \partial\Omega, \end{aligned}$$

where $\varepsilon > 0$, $\beta = (\beta_1(x), \beta_2(x))$, $f = f(x)$.

- Determine conditions in Lax-Milgram that guarantee existence and uniqueness of solution. This involves:
  - Boundedness of $F(v)$: show that $|F(v)| \le C\,\|v\|_{H^1(\Omega)}$.

– Boundedness of $a(u,v)$: show that $|a(u,v)| \leq C\,\|u\|_{H^1(\Omega)}\,\|v\|_{H^1(\Omega)}$.
– Coercivity of $a(v,v)$: (see definition 2.21) show that $a(v,v) \geq C\,\|v\|^2_{H^1(\Omega)}$.

- Prove stability estimate for u in terms of $\|f\|_{L_2(\Omega)}$, ε, $\operatorname{diam}(\Omega)$.

Solution. Weak formulation:

$$-\int_\Omega \nabla\cdot(\,\varepsilon\nabla u+\beta u)v=\int_\Omega fv \to \left\{\begin{array}{l}\text{Green's formula:}\\ \int_\Omega(\Delta u)v=\int_{\partial\Omega}(\nabla u\cdot n)v-\int_\Omega\nabla u\cdot\nabla v\end{array}\right\}\to$$

$$-\int_{\partial\Omega}(\,\varepsilon\nabla u+\beta u)\cdot n\underbrace{v}_{=0\text{ on }\partial\Omega}+\underbrace{\int_\Omega(\,\varepsilon\nabla u+\beta u)\cdot\nabla v}_{:=a(u,v)}=\underbrace{\int_\Omega fv}_{:=F(v)}$$

$$\int_\Omega(\,\varepsilon\nabla u+\beta u)\cdot\nabla v=\int_\Omega fv$$

- **Boundedness of $F(v)$:**

$$\int_\Omega fv \leq \{\text{Cauchy-Schwarz}\} \leq \|f\|_{L_2(\Omega)}\,\|v\|_{L_2(\Omega)} \leq \|f\|_{L_2(\Omega)}\,\|v\|_{H^1(\Omega)} < \infty$$

  if $f \in L_2(\Omega)$.

- **Boundedness of $a(u,v)$:**

$$\begin{aligned}|a(u,v)| &\leq \int_\Omega |\varepsilon\nabla u+\beta u|\,|\nabla v|\,\mathrm{d}x \leq \int_\Omega(\varepsilon\,|\nabla u|+|\beta|\,|u|)\,|\nabla v|\,\mathrm{d}x \leq\\ &\leq \left(\int_\Omega(\varepsilon\,|\nabla u|+|\beta|\,|u|)^2\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega|\nabla v|^2\,\mathrm{d}x\right)^{1/2} \leq\\ &\leq \max\left(\varepsilon,\|\beta\|_{L_\infty(\Omega)}\right)\left(\int_\Omega(|\nabla u|+|u|)^2\,\mathrm{d}x\right)^{1/2}\|v\|_{H^1(\Omega)} \leq\\ &\leq \left\{\ (a+b)^2 \leq 2(a^2+b^2)\ \right\} \leq\\ &\leq \max\left(\varepsilon,\|\beta\|_{L_\infty(\Omega)}\right)\left(\int_\Omega 2\left(|\nabla u|^2+|u|^2\right)\mathrm{d}x\right)^{1/2}\|v\|_{H^1(\Omega)} =\\ &= \max\left(\varepsilon,\|\beta\|_{L_\infty(\Omega)}\right)\sqrt{2}\,\|u\|_{H^1(\Omega)}\,\|v\|_{H^1(\Omega)}\,.\end{aligned}$$

  Thus, $\beta \in L_\infty(\Omega)$ gives us boundedness of $a(u,v)$.

- **Coercivity of $a(v,v)$:**

$$
\begin{aligned}
a(v,v) &= \int_\Omega (\varepsilon\,|\nabla v|^2 + v\beta\cdot\nabla v)\mathrm{d}x = \\
&= \int_\Omega (\varepsilon\,|\nabla v|^2 + v(\beta_1\partial_x v + \beta_2\partial_y v))\mathrm{d}x = \left\{\partial_x(v^2) = 2v\partial_x v\right\} = \\
&= \int_\Omega (\varepsilon\,|\nabla v|^2 + \frac{1}{2}(\beta_1\partial_x(v^2) + \beta_2\partial_y(v^2)))\mathrm{d}x = \\
&= \left\{\begin{array}{l} \text{Partial integration:} \\ \int\int \beta_1\partial_x(v^2)\mathrm{d}x\mathrm{d}y = \\ \int\left([\beta_1 v^2] - \int \partial_x\beta_1 v^2\mathrm{d}x\right)\mathrm{d}y = \\ = -\int\int \partial_x\beta_1 v^2\mathrm{d}x\mathrm{d}y \\ \text{since}\ \ v\in H_0^1(\Omega)) \end{array}\right\} = \\
&= \int_\Omega (\varepsilon\,|\nabla v|^2 - \frac{1}{2}\nabla\cdot\beta v^2)\mathrm{d}x \geq \\
&\geq \left\{\ \text{if}\ -\tfrac{1}{2}\nabla\cdot\beta \geq c > 0\ \right\} \geq \int_\Omega \varepsilon\,|\nabla v|^2 + cv^2\mathrm{d}x = \\
&= \varepsilon\,|\nabla v|^2_{H^1(\Omega)} + c\,\|v\|^2_{L_2(\Omega)} \geq \min(\varepsilon,c)\,\|v\|^2_{H^1(\Omega)}\,.
\end{aligned}
$$

So $\nabla\cdot\beta \leq -2c$ for some $c > 0$ gives coercivity.

- **Stability estimate:** We have to bound u in some norm. We note that, from above, we have

$$
\begin{aligned}
a(u,u) &= L(u) \leq c_1\,\|u\|_{H_0^1(\Omega)}\,, &\qquad c_1 &= \|f\|_{L_2(\Omega)} \\
a(u,u) &\geq c_2\,\|u\|^2_{H^1(\Omega)}\,, & c_2 &= \min(\varepsilon,c),
\end{aligned}
$$

giving us

$$
\begin{aligned}
c_2\,\|u\|^2_{H^1(\Omega)} &\leq c_1\,\|u\|_{H^1(\Omega)} \Rightarrow \\
\|u\|_{H^1(\Omega)} &\leq \frac{c_1}{c_2} = \frac{\|f\|_{L_2(\Omega)}}{\min(\varepsilon,c)}.
\end{aligned}
$$