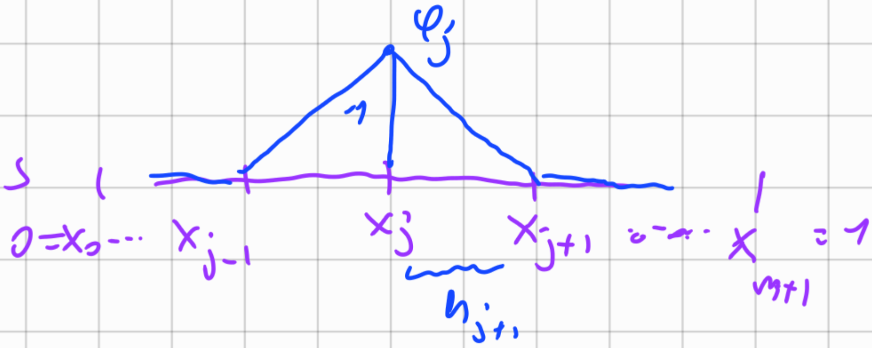
Recall:

- $L^2(0,1)$ inner product

$$(f,g)_{L^2} = \int_0^1 f(x)\,g(x)\,dx$$

- $H_0^1(0,1) = \{ v \in H^1(0,1) \; ; \; v(0) = v(1) = 0 \}$

- Hat functions



- $V_h(0,1)$ = space of continuous $\overset{\text{piecewise linear}}{\text{pw}}$ linear function on $[0,1]$

$= \text{span}\{\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_{m+1}\}$

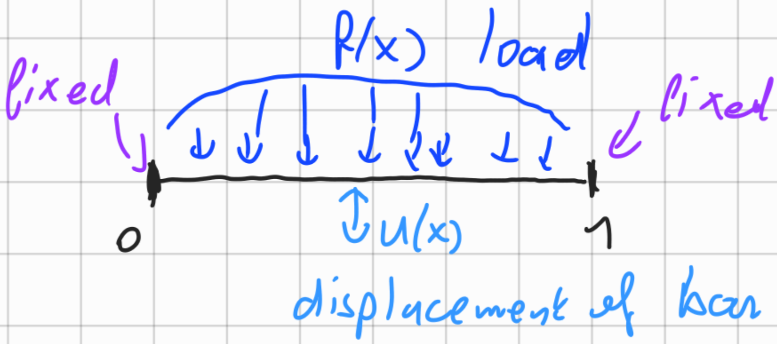
$v \in V_h(0,1) \Rightarrow v(x) = \sum_{j=0}^{m+1} \underset{v(x_j)}{\xi_j}\, \varphi_j(x)$

$\varphi_j(x) \rightarrow$ basis

$v(x_j) \rightarrow$ coordinates

2) Galerkin method:

<u>Prob:</u> Load on bar



Can be describe by eq.

$$\begin{cases} -(a(x)u'(x))' = f(x) & 0 < x < 1 \\ u(0) = 0,\ u(1) = 0 & , \end{cases}$$

where $a(x)$ is called the modulus of elasticity (given)

To simplify the presentation, take $a(x) = 1$ and get:

$$(\mathrm{BVP}) \quad \begin{cases} -u''(x) = f(x) & 0 < x < 1 & (\mathrm{DE}) \\ u(0) = 0,\ u(1) = 0\,. & & (\mathrm{BC}) \end{cases}$$

<u>Def:</u> The above BC are called

homogeneous Dirichlet BC

($= 0$ / if $= 1$ f.ex $\to$ inhomogeneous)

If f is complicated $\Rightarrow$ difficult / impossible to find the exact sol. u!

$\hookrightarrow$ need a numerical approximation $\Rightarrow$

FEM = finite element method

Idea (Galerkin):

(i) Look for a VF (done before LM)

(ii) Find a finite element solution (use $V_h(0,1)$)

(iii) Solve linear syst. of equations ("$Ax=b$")

Details:

(i) Multiply (BVP) with a test function v

(To be specified), integrate, by part:

$$\underbrace{-\int_0^1 u''(x)\,v(x)\,dx}_{\underbrace{-u'(x)\,v(x)\Big|_0^1}_{-u'(1)v(1)+u'(0)v(0)=0} + \int_0^1 u'(x)\,v'(x)\,dx \text{ (by part)}} = \int_0^1 f(x)\,v(x)\,dx \quad \text{for some } v.$$

$v(1) = 0$, $v(0) = 0$ ← force v to be zero on the boundary [homog. Dirichlet BC]

$u(0)=0, u(1)=0$

The above tells us that a good choice for a space of test functions is $H_0^1(0,1)$!!

This then gives us

(VF) Find $u \in H_0^1(0,1)$ s.t. $(u', v')_{L^2} = (f, v)_{L^2} \quad \forall v \in H_0^1(0,1)$

Rem! One can treat other type of BC.
$\rightsquigarrow$ later

(ii) Next, consider a partition of $[0,1]$, T_h:

$0 = x_0 < x_1 < x_2 \cdots < x_m < x_{m+1} = 1$, for m a

fixed integer, where $x_j - x_{j-1} = h$ for $j = 1, 2, \dots, m+1$.
same $h \to$ uniform

Consider FE space $V_h^0 = \{ v : [0,1] \to \mathbb{R} : v$ cont.
pw linear on T_h, $v(0) = 0$,
$v(1) = 0 \}$

Observe $V_h^0 \subset H_0^1$, and

$V_h^0 = \operatorname{span}(\varphi_1, \varphi_2, \dots, \varphi_m)$, with

hat functions φ_j, $j = 1, \dots, m$. $\dim(V_h^0) = m$.

With the above, we define the FE problem

(finite element problem)

(FE) Find $u_h \in V_h^0$ s.t. $(u_h', \chi')_{L^2} = (f, \chi)_{L^2} \quad \forall \chi \in V_h^0$

(iii) In order to find a linear system from

(FE), we choose $X_i = \varphi_i$ for $i = 1, 2, \dots, m$

and write $u_h(x) = \sum_{j=1}^{m} \underbrace{\xi_j}_{\text{coordinates (??)}} \cdot \underbrace{\varphi_j(x)}_{\text{basis / hat funct.}}$

We insert the above in (FE) to get:

$$\left(\sum_{j=1}^{m} \xi_j \varphi_j', \varphi_i' \right)_{L^2} = (f, \varphi_i)_{L^2} \quad \text{for } i = 1, 2, \dots, m$$

By linearity:

$$\sum_{j=1}^{m} \xi_j \underbrace{(\varphi_j', \varphi_i')_{L^2}}_{s_{ij}} = \underbrace{(f, \varphi_i)_{L^2}}_{b_i} \quad \text{for } i = 1, 2, \dots, m.$$

Finally, we get the linear system

$$S \cdot \xi = b \quad , \text{where}$$

$S = (s_{ij})_{i,j=1}^{m}$ is stiffness matrix

$b = (b_i)_{i=1}^{m}$ is load vector

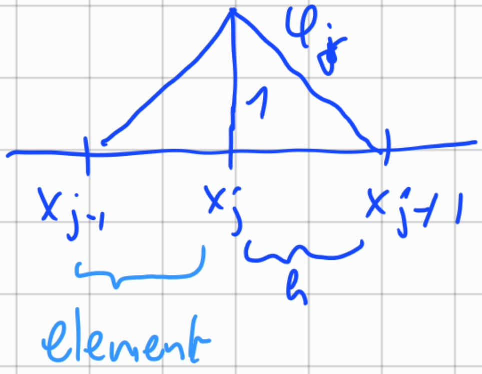
$\xi = (\xi_i)_{i=1}^{m}$ is unknown $\left(u_h(x) = \sum_j \xi_j \varphi_j(x) \right)$

Now we compute S:

F.ex. on the diagonal

$$S_{jj} = (\varphi_j', \varphi_j')_{L^2} = \int_0^1 (\varphi_j'(x))^2 \, dx$$

Recall,



$$\rightsquigarrow \varphi_j'(x) = \begin{cases} 1/h, & x_{j-1} \le x \le x_j \\ -1/h, & x_j \le x \le x_{j+1} \\ 0 & \text{else} \end{cases}$$

Hence,

$$S_{jj} = \int_0^1 (\varphi_j'(x))^2 \, dx \overset{\text{Def } \varphi_j}{=} \int_{x_{j-1}}^{x_{j+1}} (\varphi_j'(x))^2 \, dx \overset{\text{Def } \varphi_j'}{=}$$

$$= \int_{x_{j-1}}^{x_j} \left(\frac{1}{h}\right)^2 dx + \int_{x_j}^{x_{j+1}} \left(-\frac{1}{h}\right)^2 dx = \frac{1}{h^2} h + \frac{1}{h^2} \cdot h =$$

$$= \frac{2}{h}$$

$$\hookrightarrow \quad S = \frac{1}{h} \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

(at home)

For the load vector b, we get the component

$$b_j = \int_0^1 \underbrace{f(x)\varphi_j(x)}_{\to \text{ easy to integrate } \rightsquigarrow \text{ exact formula}} \, dx$$

If $f(x) \cdot \varphi_j(x)$ is complicated $\rightsquigarrow$ need to numerically approximate integrals!!

$\rightsquigarrow$ see next chapter.

Chapter V: Interpolation and numerical integration

Goal: Discuss interpolation: pass a simple functions through data points.

Discuss num. int.: $\int_0^1 f(x)\,dx \approx \dots$

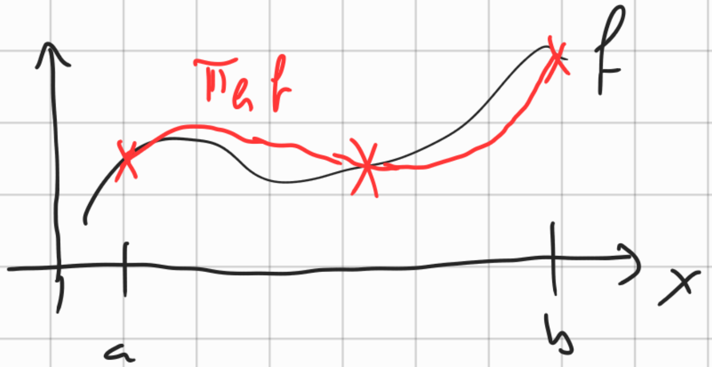
1) Polynomial interpolation:

Prob/Def: Consider a continuous function $f: [a,b] \to \mathbb{R}$ and $(q+1)$ distinct interpolation points $(x_j, f(x_j))_{j=0}^{q}$ where

$a = x_0 < x_1 < x_2 < \dots < x_q = b.$

A polynomial $\Pi_q f \in \mathcal{P}^{(q)}(a,b)$ is called a polynomial interpolant if

$$(\Pi_q f)(x_j) = f(x_j) \quad \text{for } j = 0,1,\dots,q$$



How do we compute $\Pi_q f$?

Ex! (Linear interpolation on $[0,1]$, standard basis)

linear $\rightsquigarrow$ $q=1$ $\rightsquigarrow$ 2 points $x_0=0$, $x_1=1$.

We know that $\mathcal{P}^{(1)}(0,1) = \text{span}(1,x)$ $\Rightarrow$

$\Pi_1 f(x) = c\cdot 1 + d\cdot x$, and we find c and d

using the conditions:

$$\begin{cases} \Pi_1 f(0) \stackrel{!}{=} f(0) = c\cdot 1 + d\cdot 0 & \Rightarrow c = f(0) \\ \Pi_1 f(1) \stackrel{!}{=} f(1) = c\cdot 1 + d\cdot 1 & \Rightarrow d = f(1) - f(0) \end{cases}$$

We need this equality
(by def of interpolant)

$\Rightarrow$ $\Pi_1 f(x) = f(0) + (f(1) - f(0))\cdot x$

(eq. line)

$\mathcal{P}^{(q)}(0,1) = \text{span}(1, x, x^2, \ldots, x^q)$ $\rightarrow$ $q+1$ unknow "c,d"

$(\Pi_q f)(x_j) = f(x_j)$ for $j = 0, \dots, q$ $\rightarrow$ $q+1$ equations

$\rightsquigarrow$ linear syst. "$Ax = b$"