2021-01-29 Exercise session w. 2

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1) Let $V^{(q)} = \mathcal{P}^{(q)}(0,1)$

$$V_0^{(q)} = \{ v \in V^{(q)} : v(0) = 0 \}$$

Prove that $V_0^{(q)}$ is a subspace of $V^{(q)}$

Solution.
- $(u+v)(0) = u(0) + v(0) = 0$
- $(\alpha u)(0) = \alpha u(0) = 0$
- $0 \in V_0^{(q)}$

2) Consider the IVP $\begin{cases} \dot{u}(t) = \lambda u(t) & 0 < t \le 1 \\ u(0) = u_0 \end{cases}$

Compute the Galerkin approximation for $q = 1,2,3,4$ for $\lambda = u_0 = 1$.

Solution. Variational formulation:

Find $u \in \mathcal{P}^{(q)}(0,1)$ s.t.

$$\int_0^1 \dot{u}(t)v(t) = \int_0^1 \lambda u(t)v(t)$$

$$\forall\, v \in \{v \in \mathcal{P}^{(q)} : v(0) = 0\} =: \mathcal{P}_0^{(q)}$$

Ansatz: $$u(t) = \sum_{j=0}^{q} \xi_j t^j = \{u(0) = u_0\} =$$

$$= u_0 + \sum_{j=1}^{q} \xi_j t^j$$

$$\dot{u}(t) = \sum_{j=1}^{q} j\, \xi_j\, t^{j-1}$$

$\{t^i\}_{i=1}^{q}$ spans $\mathcal{P}_0^{(q)}$

$$\Rightarrow \int_0^1 \sum_{j=1}^{q} j\, \xi_j\, t^{j-1} \cdot t^j = \int_0^1 \left(\lambda u_0 + \lambda \sum_{j=1}^{q} \xi_j t^j\right) t^i$$

$$i = 1, .., q$$

$$\sum_{j=1}^{q} \xi_j \underbrace{\left(j \int_0^1 t^{i+j-1} - \lambda \int_0^1 t^{i+j} \right)}_{a_{ij}} = \underbrace{\lambda u_0 \int_0^1 t^i}_{b_{ij}}$$

$\because A\xi = b$

$$\left.\begin{aligned} \int_0^1 t^{i+j-1} &= \left[\frac{t^{i+j}}{i+j}\right]_0^1 = \frac{1}{i+j} \\ \int_0^1 t^{i+j} &= \left[\frac{t^{i+j+1}}{i+j+1}\right]_0^1 = \frac{1}{i+j+1} \end{aligned}\right\} a_{ij} = \frac{j}{i+j} - \frac{\lambda}{i+j+1}$$

$$\int_0^1 t^i = \frac{1}{i+1} \Rightarrow b_i = \frac{\lambda u_a}{i+1}$$

$\underline{q=1:}$

$$a_{11} = \frac{1}{1+1} - \frac{1}{1+1+1} = \frac{1}{6}$$

$$b_1 = \frac{1 \cdot 1}{1+1} = \frac{1}{2}$$

$$\Rightarrow \text{lin. sys.:} \quad \frac{1}{6}\xi_1 = \frac{1}{2} \Rightarrow \xi_1 = 3$$

$$\underline{u_1(t) = 1 + 3t}$$

$\underline{q=2:}$ $A = \ldots = \begin{bmatrix} 1/6 & 5/12 \\ 1/12 & 3/10 \end{bmatrix}$, $b = \ldots = \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix}$

$A\xi = b \Rightarrow \xi = \begin{bmatrix} 8/11 \\ 10/11 \end{bmatrix}$

$$\Rightarrow \underline{u_2(t) = 1 + \frac{8}{11} t + \frac{10}{11} t^2}$$

$\underline{q=3}$ $A = \ldots = \begin{bmatrix} \frac{1}{6} & \frac{5}{12} & \frac{11}{20} \\ \frac{1}{12} & \frac{3}{10} & \frac{13}{30} \\ \frac{1}{20} & \frac{7}{30} & \frac{5}{14} \end{bmatrix}$ $b = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$

$A\xi = b \Rightarrow \xi = \begin{bmatrix} \frac{30}{29} & \frac{45}{116} & \frac{35}{116} \end{bmatrix}^T$

$$\underline{u_3(t) = 1 + \frac{30}{29} t + \frac{45}{116} t^2 + \frac{35}{116} t^3}$$

$q=4$:

$$A = \begin{bmatrix} \frac{1}{6} & \frac{5}{12} & \frac{11}{20} & \frac{19}{30} \\ \frac{1}{12} & \frac{3}{10} & \frac{13}{30} & \frac{11}{21} \\ \frac{1}{20} & \frac{7}{30} & \frac{5}{14} & \frac{25}{26} \\ \frac{1}{30} & \frac{8}{42} & \frac{17}{56} & \frac{7}{18} \end{bmatrix} \quad b = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/4 \\ 1/5 \end{bmatrix}$$

$$\Rightarrow U_4(t) = 1 + \frac{1704}{1709} t + \frac{882}{1709} t^2 + \frac{224}{1709} t^3 + \frac{126}{1709} t^4.$$

3) Compute the $L_2(0,1)$ projection into $\mathcal{P}^3(0,1)$ of the exact solution u for

$$\begin{cases} \dot{u}(t) = \lambda u(t) & 0 < t \leq 1 \\ u(0) = u_0 \end{cases}$$

$\lambda = u_0 = 1$, Compare with Galerkin solution.

Solution. Exact sol: $u = e^t$

The $L_2(0,1)$ projection $\tilde{u} \in \mathcal{P}^3(0,1)$ of u satisfies

$$\int_0^1 \tilde{u}(t) v(t)\, dt = \int_0^1 u(t) v(t)\, dt \quad \forall v \in \mathcal{P}^3(0,1)$$

Ansatz: $\tilde{u}(t) = \sum_{j=0}^{3} \xi_j t^j$. Let $v(t) = t^i$
$i = 0, \dots, 3$

$$\Rightarrow \sum_{j=0}^{3} \xi_j \underbrace{\int_0^1 t^j t^i\, dt}_{a_{ij}} = \underbrace{\int_0^1 e^t t^i\, dt}_{b_i}$$

$$a_{ij} = \int_0^1 t^{i+j}\, dt = \frac{1}{i+j+1} \qquad b_0 = \int_0^1 e^t\, dt = e - 1$$

$$b_1 = \int_0^1 e^t t\, dt = \{P.I.\} = 1$$

$$b_2 = \dots = e - 2$$

$$b_3 = \dots = -2e + 6$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} \tilde{\xi}_0 \\ \tilde{\xi}_1 \\ \tilde{\xi}_2 \\ \tilde{\xi}_3 \end{bmatrix} = \begin{bmatrix} e-1 \\ 1 \\ e-2 \\ 6-2e \end{bmatrix}$$

$$\Rightarrow \tilde{u}(t) \approx 0{,}991 + 1{,}0183t + 0{,}4212t^2 + 0{,}2786\,t^3$$

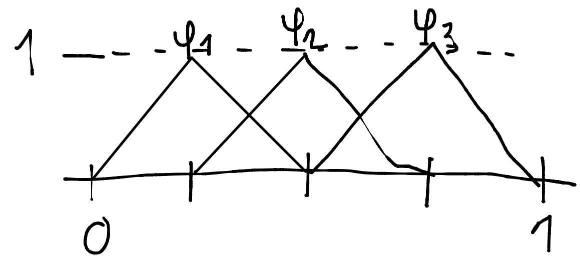
Galerkin: $1 + 1{,}0345\,t + 0{,}3879\,t^2 + 0{,}3017\,t^3$

4) Consider the BVP

$$\begin{cases} -(a(x)u'(x))' = f(x) & 0 < x \leq 1 \\ u(0) = u(1) = 0 \end{cases}$$

$a \equiv 1$, $f = x$. Let $h = \frac{1}{4}$ and compute Galerkin approximation cG(1) for u.

Solution.



$$\varphi_1(x) = \begin{cases} 4x, & 0 \le x \le \frac{1}{4} \\ 4\left(\frac{1}{2} - x\right), & \frac{1}{4} \le x \le \frac{1}{2} \end{cases}$$

$$\varphi_2(x) = \begin{cases} 4\left(x - \frac{1}{4}\right) & \frac{1}{4} \le x \le \frac{1}{2} \\ 4\left(\frac{3}{4} - x\right), & \frac{1}{2} \le x \le \frac{3}{4} \end{cases}$$

$$\varphi_3(x) = \begin{cases} 4\left(x - \frac{1}{2}\right), & \frac{1}{2} \le x \le \frac{3}{4} \\ 4(1 - x), & \frac{3}{4} \le x \le 1 \end{cases}$$

DE: $-u'' = x$. Multiply DE by $v \in \{\varphi_1, \varphi_2, \varphi_3\}$ and integrate

$$-\int u''v = \int xv \quad \text{P.I.} \Rightarrow \int u'v' = \int xv$$

Let $u_h = \sum_{i=1}^{3} \xi_i \varphi_i(x)$

$$\Rightarrow \sum_{i=1}^{3} \xi_i \underbrace{\int_0^1 \varphi_i' \varphi_j'}_{a_{ji}} = \underbrace{\int_0^1 x \varphi_j}_{b_j}, \quad j = 1,2,3$$

$$\left.\begin{array}{l} a_{jj} = \int_{x_{j-1}}^{x_{j+1}} (\varphi_j'(x))^2 \, dx = \frac{2}{h} = 8 \\ a_{j,j+1} = \ldots = -\frac{1}{h} = -4 \overset{\text{symm.}}{=} a_{j+1,j} \\ a_{i,j} \text{ with } |i-j| \geq 2 = 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow A = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}$$

$$b_j = \int_0^1 \varphi_j(x) \cdot x \, dx = \int_{x_{i-1}}^{x_i} \frac{x - x_{i-1}}{h} \cdot x \, dx +$$

$$+\int_{x_i}^{x_{i+1}} \frac{x_{i+1}-x}{h}\, x\, dx = \ldots = h^2 j = \frac{j}{16}$$

$$\Rightarrow b = \left[\frac{1}{16} \;\; \frac{1}{8} \;\; \frac{3}{16}\right]^T$$

Solving $A\xi = b$ gives us $\xi = \left[\frac{5}{128} \;\; \frac{1}{16} \;\; \frac{7}{128}\right]^T$

$\Rightarrow$ Galerkin approx. $cG(1)$ is:

$$u_h(x) = \frac{5}{128}\varphi_1(x) + \frac{1}{16}\varphi_2(x) + \frac{7}{128}\varphi_3(x)$$

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5) $V^{(q)} = \operatorname{span}\{\sin(i\pi x),\ 1 \leq i \leq q\}$

Prove that $V^{(q)}$ is a subspace of $C_0([0,1]) = \{f \in C([0,1]) : f(0) = f(1) = 0\}$

Show that $\{\sin(i\pi x)\}_{i=1}^{q}$ forms an orthogonal basis of $V^{(q)}$!

<u>Solution.</u> Obviously $V^{(q)} < C_0([0,1])$

$f, g \in V^{(q)} \Rightarrow \alpha f + \beta g \in V^{(q)} \quad \forall \alpha, \beta \in \mathbb{R}$

For <u>basis</u>, we need linear independence

but lin. indep. follows from orthog.

Suffices to show orthogonality.

$$\int_0^1 \sin(i\pi x)\sin(j\pi x)\,dx =$$

$$= \left[\frac{-\cos(i\pi x)}{i\pi}\sin(j\pi x)\right]_0^1 + \int_0^1 \frac{\cos(i\pi x)}{i\pi}\cos(j\pi x)\, j\pi\,dx$$

$$= \frac{j}{i}\int_0^1 \cos(i\pi x)\cos(j\pi x)\,dx =$$

$$= \frac{j}{i}\left[\frac{\sin(i\pi x)}{i\pi}\cos(j\pi x)\right]_0^1 +$$

$$+\frac{j}{i}\int_0^1 \frac{\sin(i\pi x)}{i\pi}\, j\pi \sin(j\pi x)\,dx$$

$$=\frac{j^2}{i^2}\int_0^1 \sin(i\pi x)\sin(j\pi x)\,dx$$

$$\Rightarrow \underbrace{\left(1-\frac{j^2}{i^2}\right)}_{\neq 0,\ j\neq i}\int_0^1 \sin(i\pi x)\sin(j\pi x)\,dx = 0$$

$$\Rightarrow \int_0^1 \sin(i\pi x)\sin(j\pi x)\,dx = 0$$

thus orthogonal and linearly indep. //.

3.5) Find approximate solution $U(x)$ to

$$\begin{cases} -u''(x) = 1, & 0<x<1 \\ u(0) = u(1) = 0 \end{cases}$$

using the Ansatz $U(x) = A\sin(\pi x) + B\sin(2\pi x)$

a) Calculate exact solution

b) Write down residual $R(x) = -U''(x) - 1$

c) Use the orthogonality condition

$$\int_0^1 R(x) \sin(\pi n x)\, dx = 0 \qquad n \in \{1, 2\}$$

to determine A and B.

d) Plot the error $e(x) = u(x) - U(x)$

Solution.

a) $-u'' = 1$

$$u' = -x + a$$

$$u = \frac{-x^2}{2} + ax + b$$

$$u(0) = 0 \Rightarrow b = 0 \Rightarrow u = \frac{-x^2}{2} + ax$$

$$u(1) = 0 \Rightarrow a = \frac{1}{2} \Rightarrow \underline{u = \frac{-x^2}{2} + \frac{x}{2}}$$

b) $U(x) = A \sin(\pi x) + B \sin(2\pi x)$

$$u'(x) = A\pi\cos(\pi x) + 2B\pi\cos(2\pi x)$$

$$u''(x) = -A\pi^2\sin(\pi x) - 4B\pi^2\sin(2\pi x)$$

$$\Rightarrow R(x) = A\pi^2\sin(\pi x) + 4B\pi^2\sin(2\pi x) - 1$$

c) $\int_0^1 \left(A\pi^2\sin(\pi x) + 4B\pi^2\sin(2\pi x) - 1\right)\sin(n\pi x)\,dx = 0$

$\quad = \text{I}$

$n = 1, 2$

$\underline{n=1}$ $\text{I} = \{\perp\} = \int_0^1 \left(A\pi^2\sin^2(\pi x) - \sin(\pi x)\right)dx = 0$

$$A\pi^2\int_0^1 \sin^2(\pi x)\,dx = \int_0^1 \sin(\pi x)\,dx$$

$$\Rightarrow \frac{A\pi^2}{2} = \frac{2}{\pi} \Rightarrow \underline{A = \frac{4}{\pi^3}}$$

$\underline{n=2}$: $\text{I} = \{\perp\} =$

1

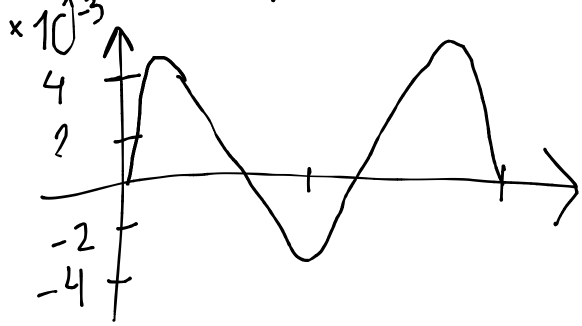
$$= \int_0^1 \left(4B\pi_1^2 \sin(2\pi x) - 1\right) \sin(2\pi x)\,dx = 0$$

$$4B\pi^2 \int_0^1 \sin^2(2\pi x) = \int_0^1 \sin(2\pi x)$$

$$\ldots \Rightarrow \frac{4B\pi^2}{2} = 0 \Rightarrow \underline{B=0}$$

$$\underline{U(x) = \frac{4}{\pi^3}\sin(\pi x)}$$

d) Plot of error:



3.7) $U(x) = \xi_0 \phi_0(x) + \xi_1 \phi_1(x)$ be an approximate sol. to

$$\begin{cases} -u''(x) = x - 1 & 0 < x < \pi \\ u'(0) = u(\pi) = 0 \end{cases}$$

$$\phi_0(x) = \cos\frac{x}{2} \qquad \phi_1 = \cos\frac{3x}{2}$$

a) Find analytical solution.

$$u'' = 1 - x$$

$$u' = x - \frac{x^2}{2} + a$$

$$u = \frac{x^2}{2} - \frac{x^3}{6} + ax + b$$

$$u'(0) = 0 \Rightarrow a = 0$$

$$u(\pi) = 0 \Rightarrow b = \frac{\pi^3}{6} - \frac{\pi^2}{2} = \frac{\pi^3 - 3\pi^2}{6}$$

$$\therefore\ u(x) = \frac{x^2}{2} - \frac{x^3}{6} + \frac{\pi^3 - 3\pi^2}{6}$$

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b) $R(x) = u'' + x - 1$

$$u'' = -\frac{\xi_0}{4}\cos\frac{x}{2} - \frac{9\xi_1}{4}\cos\frac{3x}{2}$$