Recall: • FE of Poisson's eq: $\begin{cases} -u''(x) = f(x) \\ u(0)=0,\ u(1)=0 \end{cases}$

Find $u_h \in V_h$ s.t. $(u_h', \chi') = (f, \chi)$ $\forall \chi \in V_h$

where $V_h = \{ v : [0,1] \to \mathbb{R} : v \text{ cont. pw linear on } \mathcal{T}_h,\ v(0)=0,\ v(1)=0 \}$

$= \operatorname{span}(\varphi_1, \dots, \varphi_m)$ φ_j hat fct

gives $S\xi = b$, $b = (b_j)_{j=1}^{m}$, $b_j = \int_0^1 f(x)\varphi_j(x)\,dx$

$\int\int$??

• Interpolation:

Use $P^{(q)}(a,b) = \operatorname{span}(1, x, x^2, \dots, x^q)$

$\downarrow$

polyn. of degree $\le q$

Consider $(q+1)$ distinct points

$$a = x_0 < x_1 < x_2 < \dots < x_q = b$$

Def: Lagrange polynomials are defined as

$$\lambda_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{q} \frac{x - x_j}{x_i - x_j} \quad \text{for } i = 0, 1, 2, \dots, q$$

Rem:
- $\lambda_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ $\rightsquigarrow$ nodal basis
- $\mathcal{P}^{(q)}(a,b) = \text{span}(\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_q)$.

Ex: linear interpolation on $[0,1]$ (Lagrange basis)

linear $\Rightarrow$ $q = 1$ and $x_0 = 0 < x_1 = 1$.

Use Lagrange polynomials

$$\lambda_0(x) \overset{\text{Def } \lambda_i}{=} \frac{x - x_1}{x_0 - x_1} \overset{\text{Def } x_0, x_1}{=} \frac{x - 1}{-1} = 1 - x$$

$$\lambda_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x}{1} = x$$

We know that $\mathcal{P}^{(1)}(0,1) = \text{span}(\lambda_0, \lambda_1) =$
$= \text{span}(1 - x, x)$.

In order to find the linear interpolant

$\Pi_1 f$ of some given function f, we observe

$$\Pi_1 f(x) = c_0 \cdot \lambda_0(x) + c_1 \cdot \lambda_1(x) = c_0(1-x) + c_1 \cdot x$$

λ_i basis

$\Pi_1 f$ should interpolate f

$$\Pi_1 f(x_0) \stackrel{!}{=} f(x_0) \stackrel{x_0=0}{\Longleftrightarrow} c_0 = f(x_0)$$

$$\Pi_1 f(x_1) \stackrel{!}{=} f(x_1) \stackrel{x_1=1}{\Longleftrightarrow} c_1 = f(x_1)$$

$$\leadsto \Pi_1 f(x) = f(x_0) \cdot (1-x) + f(x_1) \cdot x =$$

$$= f(x_0) + x \cdot (f(x_1) - f(x_0))$$

(eq. line, same as last week)

Ex: $q=2 \leadsto$ book p. 92, expl. 3.7

What is the error of such polynomial interpolant?

Recall: How to measure distances between functions?

Use L^p-norms!

$$\|f\|_{L^p(a,b)} = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p} \qquad p = 1,2$$

$$\|f\|_{L^\infty(a,b)} = \max_{x \in [a,b]} |f(x)| \qquad (f \text{ cont.})$$

"Distance" between f and g : $\|f-g\|_{L^{\cdots}}$

<u>Th</u>: Let $p = 1, 2, \infty$. Assume $f \in C^2(a,b)$ and $f', f'' \in L^p(a,b)$. Then, $\exists$ constants c_1, c_2, c_3 s.t.

1) $\underbrace{\|\Pi_1 f - f\|_{L^p(a,b)}}_{\text{error}} \leq C_1 \cdot (b-a)^2 \cdot \|f''\|_{L^p(a,b)}$

2) $\|\Pi_1 f - f\|_{L^p(a,b)} \leq C_2 \cdot (b-a) \cdot \|f'\|_{L^p(a,b)}$

3) $\|(\Pi_1 f)' - f'\|_{L^p(a,b)} \leq C_3 \cdot (b-a) \cdot \|f''\|_{L^p(a,b)}$

This for linear interpolation $\Pi_1 f$.

<u>Proof</u> (of 1) and $p = \infty$)

- From previous example, we know

$$\Pi_1 f(x) = f(a) \cdot \lambda_0(x) + f(b) \cdot \lambda_1(x) \quad , \text{ where}$$

We have used Lagrange polynomials

$$\lambda_0(x) = \frac{x-b}{a-b}, \quad \lambda_1(x) = \frac{x-a}{b-a}$$

- Using a Taylor expansion, we get

$$f(a) = f(x) + f'(x)(a-x) + \frac{f''(\xi_0)}{2}(a-x)^2, \text{ where } \xi_0 \in (a,x)$$

$$f(b) = f(x) + f'(x)(b-x) + \frac{f''(\xi_1)}{2}(b-x)^2, \text{ where } \xi_1 \in (x,b).$$

- The above gives us

$$\Pi_1 f(x) = f(a)\lambda_0(x) + f(b)\lambda_1(x) \overset{f(a),f(b)}{=} \dots =$$

$$= f(x) + \frac{1}{2} f''(\xi_0)(a-x)^2 \lambda_0(x) + \frac{1}{2} f''(\xi_1)(b-x)^2 \lambda_1(x)$$

- For the error, we thus get

$$|\Pi_1 f(x) - f(x)| \overset{\Delta}{\le} \frac{1}{2} |f''(\xi_0)| |a-x|^2 |\lambda_0(x)| +$$

$$+ \frac{1}{2} \underbrace{|f''(\xi_1)|}_{\le \max_{y\in[a,b]} |f''(y)|} \underbrace{|b-x|^2}_{\le (b-a)^2 \text{ (since } x\in(a,b))} \cdot \underbrace{|\lambda_1(x)|}_{\le 1 \text{ (def } \lambda_1)}.$$

$$\leq \max_{y \in [a,b]} |f''(y)| \cdot (b-a)^2$$

$$\Rightarrow \|\Pi_1 f - f\|_{L^\infty} \leq \|f''\|_{L^\infty} \cdot (b-a)^2 \qquad \blacksquare$$

(using def L^∞-norm)

2) Continuous piecewise linear interpolation:

We do the "same" as above with pw linear functions (used for instance in FEM).

Recall: $V_h(a,b) = \{ v : [a,b] \to \mathbb{R} : v$ is cont. pw linear on $T_h \} =$

$$= \text{Span}(\varphi_0, \varphi_1, \dots, \varphi_{m+1}),$$

where T_h is a partition of $[a,b]$:

$a = x_0 < x_1 < \dots < x_m = x_{m+1} = b$, and

$h_j = x_j - x_{j-1}$, and φ_j are hat functions

Any $v \in V_h$ can be written as

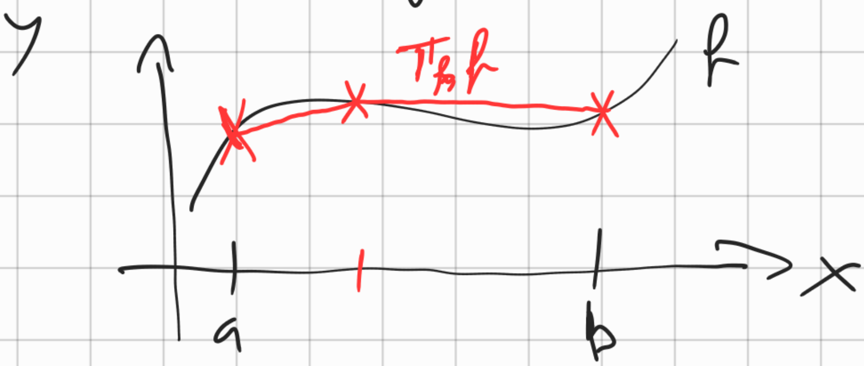
$$v(x) = \sum_{j=0}^{m+1} v(x_j) \varphi_j(x)$$

Def: A mesh function is defined as

$$h(x) = h_j \quad \text{for } x \in (x_{j-1}, x_j)$$

Def: Let $f : [a,b] \to \mathbb{R}$ be continuous and the above partition T_h. The continuous piecewise linear interpolant of f is defined by

$$\Pi_h f(x) = \sum_{j=0}^{m+1} f(x_j)\, \varphi_j(x) \quad \text{for } x \in [a,b]$$



Ex: Let $f(x) = (x-1)(x-2)+1$ for $x \in [1,3]$ and uniform partition of $[1,3]$ into 2 subintervals. Find $\Pi_h f$?

We have $m = 1$ and $T_h: \; x_0 < x_1 < x_2$

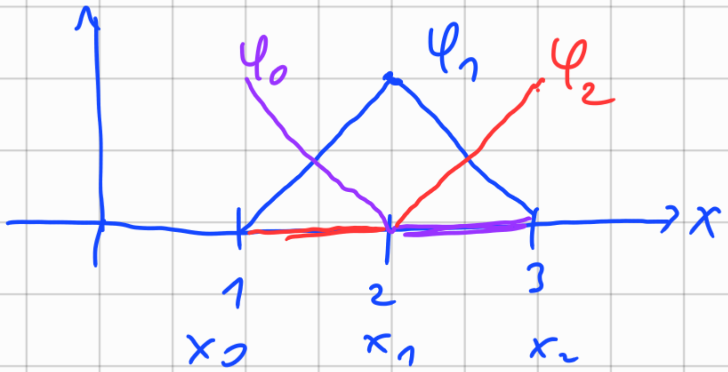
$$\begin{array}{ccccc} x_0 & < & x_1 & < & x_2 \\ \shortparallel & & \shortparallel & & \shortparallel \\ 1 & & 2 & & 3 \end{array}$$

and $h = \frac{b-a}{m+1} = \frac{3-1}{2} = 1.$

By def, $\Pi_h f(x) = f(x_0)\cdot \varphi_0(x) + f(x_1)\varphi_1(x) + f(x_2)\varphi_2(x)$,

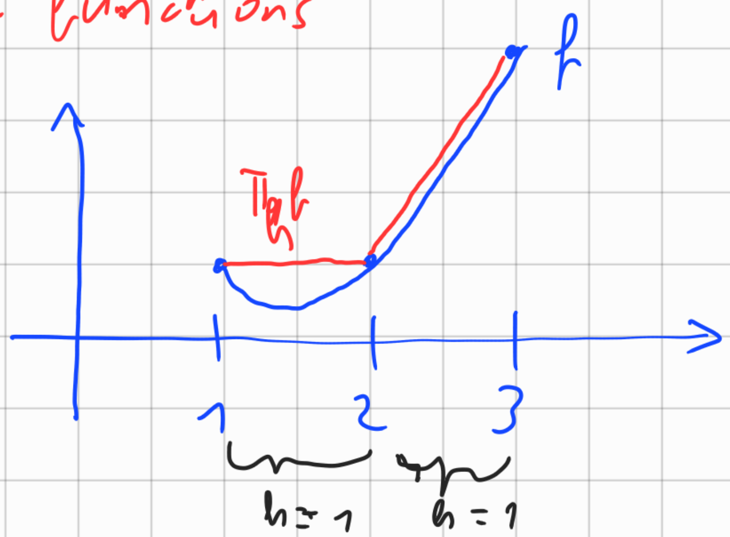
where $f(x_0) = f(1) = 1$, $f(x_1) = f(2) = 1$,

$f(x_2) = f(3) = 3$



This gives us: $\Pi_1 f(x) = \ldots = \varphi_0(x) + \varphi_1(x) + 3\varphi_2(x) =$

$$\underset{\text{use def } \varphi_j \text{ hat functions}}{\overset{*}{=}} \begin{cases} \frac{x-x_1}{-h} + \frac{x-x_0}{h} + 0 = 1 & \text{for } x_0 \leq x \leq x_1 \\ 0 + \frac{x-x_2}{-h} + 3\left(\frac{x-x_1}{h}\right) = 2x-3 & \text{for } x_1 \leq x \leq x_2. \end{cases}$$



It seems $h \to 0$ then $\Pi_h f$ better approx to f.

Th: Let $f \in C^2(a,b)$ and $\Pi_h f$ cont pw lin. interp.

Then, the errors can be bounded by

1) $\|\Pi_h f - f\|_{L^p} \leq C_1 \cdot \boxed{h^2} \cdot \|f''\|_{L^p}$

2) $\|\Pi_h f - f\|_{L^p} \leq C_2 \cdot h \cdot \|f'\|_{L^p}$

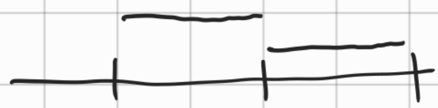
3) $\|(\Pi_h f)' - f'\|_{L^p} \leq C_3 \cdot h \; \|f''\|_{L^p}$

for $p = 1, 2, \infty$ and a uniform partition.

Rem: • error $\to 0$ as $h \to 0$

• error is larger when f'' is large (i.e. f bends a lot)

• for non-uniform partition, $h(x)$ mesh function, 1) reads



1) $\|\Pi_h f - f\|_{L^p} \leq C \cdot \|h^2 f''\|_{L^p}$

und similar for other estimates.

Proof: (1) and $p = 1, 2$, uniform grid)

$$\|\Pi_h f - f\|^p_{L^p(a,b)} \underset{\text{Def norm}}{=} \int_a^b |\Pi_h f(x) - f(x)|^p \, dx =$$

$$= \sum_{j=1}^{m+1} \int_{x_{j-1}}^{x_j} |\Pi_h f(x) - f(x)|^p \, dx = \sum_{j=1}^{m+1} \|\Pi_h f - f\|^p_{L^p(x_{j-1}, x_j)}$$

Def Π_h Def norm linear by def $\Pi_h f$

$$\leq \sum_{j=1}^{m+1} \left(C \cdot h^2\right)^p \|f''\|^p_{L^p(x_{j-1}, x_j)} \leq (C h^2)^p \sum_{j=1}^{m+1} \|f''\|^p_{L^p(x_{j-1}, x_j)}$$

Previous Theorem

$$\leq (C h^2)^p \, \|f''\|^p_{L^p(a,b)}$$

$\Rightarrow$ Error $\|\Pi_h f - f\|_{L^p(a,b)} \leq C \cdot h^2 \|f''\|_{L^p(a,b)}$ $\blacksquare$

3) Numerical integration / quadrature formulas

Prob: Find approximations of integrals $\int_a^b f(x) dx$

Idea: Approximate $f(x) \approx$ polyn. of degree q $p(x)$

$$\Rightarrow \int_a^b f(x) dx \approx \underbrace{\int_a^b p(x) dx}_{\text{easy to integrate!}}$$

The following examples are classical

quadrature formulas:

- $q=0$: $f(x) \approx f\left(\frac{a+b}{2}\right)$ (constant polyn.)

Then, $\int_a^b f(x)\,dx \approx \int_a^b f\left(\frac{a+b}{2}\right)dx = (b-a)\, f\left(\frac{a+b}{2}\right)$

This gives us the midpoint rule

$$\int_a^b f(x)\,dx \approx (b-a)\cdot f\left(\frac{a+b}{2}\right)$$