Recall: • BVP, ex., $\begin{cases} -u''(x) = f(x) \\ u(0)=0,\ u(1)=0 \end{cases}$

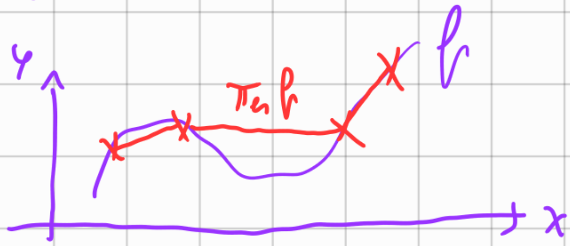
(FE) Find $u_h \in V_h^0$ s.t. ... blabla ...

$V_h^0 = \{ v : [0,1] \to \mathbb{R} : v \text{ cont pw linear}^{T_h},\ v(0)=0,\ v(1)=0 \}$

$= \operatorname{span}(\varphi_1, \varphi_2, \dots, \varphi_m)$ $\qquad \varphi_j$ hat

• Cont pw linear interpolant of f:

$$\underset{\underset{V_h}{\cap}}{\Pi_h f}(x) = \sum_{j=0}^{m+1} f(x_j)\,\varphi_j(x)$$



$$\| \Pi_h f - f \|_{L^p} \le C \cdot \| h^2 f'' \|_{L^p}$$

$$\| (\Pi_h f)' - f' \|_{L^p} \le C \cdot \| h f'' \|_{L^p}$$

• FE gives linear syst. "$Ax = b$" $\rightsquigarrow$ contains integrals

• Quadrature formulas $\approx \int_a^b f(x)\,dx$

Midpoint rule $\int_a^b f(x)\,dx \approx (b-a)\, f\left(\frac{a+b}{2}\right)$

(ii) $q = 1$, Lagrange interpolant through points $x_0 = a$, $x_1 = b$, we get

$$f(x) \approx \Pi_1 f(x) = f(a)\lambda_0(x) + f(b)\lambda_1(x) =$$

$$= f(a)\frac{x-b}{a-b} + f(b)\frac{x-a}{b-a}$$

$$\int_a^b f(x)\,dx \approx \int_a^b \Pi_1 f(x)\,dx = \frac{f(a)}{a-b}\int_a^b (x-b)\,dx +$$

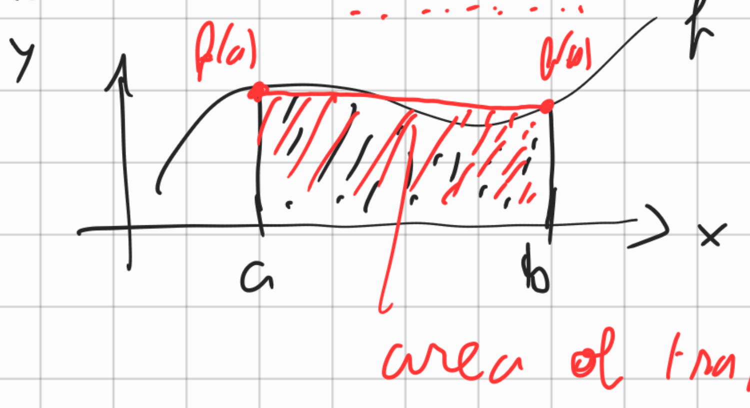
$$+ \frac{f(b)}{b-a}\int_a^b (x-a)\,dx = \frac{f(a)}{\cancel{a-b}}\left(-\frac{(a-b)^{\cancel{2}}}{2}\right)$$

$$+ \frac{f(b)}{\cancel{b-a}}\frac{(b-a)^{\cancel{2}}}{2} = \frac{f(a)}{2}(b-a) + \frac{f(b)}{2}(b-a)$$

$$= \frac{b-a}{2}\left(f(a) + f(b)\right)$$

$\hookrightarrow$ This gives us the trapezoidal rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left(f(a)+f(b)\right)$$



(iii) $q=2$, Lagrange interpolant through a, $\frac{a+b}{2}$, b

... One gets Simpson's rule :

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

Ex: Use the above quadrature formulas to find approximations of $\int_0^{\pi/4} \sin(x)\,dx$:

exact value $\int_0^{\pi/4} \sin(x)\,dx \approx 0{,}29289...$

Midpoint rule $\int_0^{\pi/4} \sin(x)\,dx \approx \left(\frac{\pi}{4} - 0\right) \sin\left(\frac{\pi/4+0}{2}\right)$

$\approx 0{,}3...$

Trapezoidal rule $\int_0^{\pi/4} \sin(x)\,dx \approx \frac{\pi/4}{2}\left(\sin\left(\frac{\pi}{4}\right) + \sin(0)\right)$

$\approx 0{,}27$

Simpson's rule $\int_0^{\pi/4} \sin(x)\,dx \approx \dots \approx 0{,}2929$

⚠ In practice, one consider a partition of $[a,b]$: $a = x_0 < x_1 < \dots < x_N = b$, $x_j - x_{j-1} = h$

and then

$$\int_a^b f(x)\,dx = \sum_{j=1}^{N} \underbrace{\int_{x_{j-1}}^{x_j} f(x)\,dx}_{\text{apply quadrature formula here!}} \approx \sum_{j=1}^{N} QF(f, x_j, x_{j-1})$$

f.ex midpoint

⚠ $(x_{j-1}, x_j]$ small

Chapter VI : FEM for BVP

Goal: Present and analyse FEM for several BVP.

1) Model problems

Consider BVP

$$(BVP)\quad \begin{cases} -(a(x)u'(x))' = f(x) & 0 < x < 1 \\ u(0) = 0,\ u(1) = 0, \end{cases}$$

where $a > 0$ (a pw cont, bounded, f cont + bounded) $a(x) \geq a_0 > 0$

In order to give a VF of BVP, we consider $H_0^1 = \{ v : [0,1] \to \mathbb{R} : v \in H^1,\ v(0) = 0,\ v(1) = 0 \}$

and get

$$-\int_0^1 (a(x)u'(x))'\,v(x)\,dx = \underbrace{-(a(x)u'(x))v(x)\Big|_0^1}_{=0 \text{ since } v \in H_0^1} + \int_0^1 a(x)u'v'$$

(VF) Find $u \in H_0^1$ s.t.

$$\int_0^1 a(x)u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx \quad \forall v \in H_0^1$$

To get FE problem, we consider partition of $[0,1]$: T_h: $0 = x_0 < x_1 < x_2 < \dots < x_m < x_{m+1} = 1$, where $x_j - x_{j-1} = h_j$. Set mesh function $h(x) = h_j$ for $x \in (x_{j-1}, x_j)$ and $j = 1, 2, \dots, m+1$.

Consider $V_h^0 = \text{span}(\varphi_1, \varphi_2, \dots, \varphi_m)$, φ_j hat fct.

This gives FE problem:

(FE) Find $u_h \in V_h^0$ s.t.

$$\int_0^1 a(x)\, u_h'(x)\, v'(x)\, dx = \int_0^1 f(x)\, v(x)\, dx \quad \forall v \in V_h^0$$

The above is called cG(1) FEM

continuous Galerkin linear (deg = 1)

[assign. cG(2) quadratic (deg = 2)]

Observation: $V_h^0 \subset H_0^1$, this means that we can can also take test functions $v \in V_h^0$ in VF!

"VF - FE" $\rightsquigarrow$ $\int_0^1 a(x)\left(u'(x) - u_h'(x)\right) v'(x)\, dx = 0 \quad \forall v \in V_h^0$

(GO)

This relation is called Galerkin orthogonality.

GO means that the error of FEM $u-u_h$ is orthogonal in V_h^0 for the energy inner product, that we now define

<u>Def</u>: For $f,g \in H_0^1$, and "a" as above, one defines

- L_a^2-inner product / weighted inner product

$$(f,g)_a = \int_0^1 a(x) f(x) g(x) dx$$

- Energy inner product

$$(f,g)_E = (f',g')_a = \int_0^1 a(x) f'(x) g'(x) dx$$

- With their norms

$$\|f\|_a = \sqrt{(f,f)_a} \quad , \quad \|f\|_E = \sqrt{(f,f)_E}$$

2) A priori error estimates in energy norm:

A priori means "error $\leq C(h, u)$"
exact sol.

We first show that the FE solution u_h

is the best approximation of u in V_h^0 in the energy norm.

<u>Th</u>: Let u be sol. to (VF), u_h be FE sol.

Then, one has:

$$\|u - u_h\|_E \le \|u - v\|_E \quad \forall v \in V_h^0.$$

<u>Proof</u>:

Consider $\|u - u_h\|_E^2 \underset{\text{Def } \|\cdot\|_E}{=} (u - u_h, u - u_h)_E \underset{\text{Def } (\cdot,\cdot)_E}{=} (u' - u_h', u' - u_h')_a =$

$= (u' - u_h', u' - v' + v' - u_h')_a \underset{\text{linearity}}{=} (u' - u_h', u' - v')_a +$

for some $v \in V_h^0$

$+ \underbrace{(u' - u_h', v' - u_h')_a}_{= 0 \text{ by (GO) since } v, u_h \in V_h^0} = (u' - u_h', u' - v')_a \overset{C-S}{\le}$

$\le \|u' - u_h'\|_a \cdot \|u' - v'\|_a \underset{\text{Def } \|\cdot\|_E}{=} \|u - u_h\|_E \cdot \|u - v\|_E$

$\Rightarrow \|u - u_h\|_E \le \|u - v\|_E \quad \forall v \in V_h^0$ $\blacksquare$

$$\|u-u_h\|_E^{\cancel{2}} \le \|u-\cancel{u_h}\|_E \cdot \|u-v\|_E$$

Using the above result, we get:

<u>Th:</u> (a priori error estimate) (Model problem)

Let u be sol. (VF), let u_h be sol. (FE).

Assume $u'' \in L^2_a(0,1)$. Then, one has the error estimate:

$$\|u-u_h\|_E \le C \cdot \|h u''\|_a \quad \left(\xrightarrow[h(x)\to 0]{} 0 \right)$$

<u>Rem:</u> • $h = h(x)$ mesh fct.

If uniform mesh $h(x) = \hat{h}$, $\hat{h} = \frac{1}{n+1}$, then

$$\text{error} \le C \cdot \hat{h} \, \|u''\|_a \xrightarrow[\hat{h}\to 0]{} 0$$

• Not optimal; $\|u-u_h\|_{L^2(0,1)} \le C \cdot h^2 \|u\|_{H^2}$

<u>Proof:</u> Let us start with

$$\|u-u_h\|_E \le \|u-\Pi_h u\|_E = \|u' - (\Pi_h u)'\|_a =$$

Above theorem, where $\Pi_h u \in V_h^0$ is cont. pw linear interpolant of u

Def $\|\cdot\|_E$

$$= \left(\int_0^1 \underbrace{a(x)}_{\le \max\limits_{y\in[0,1]}(a(y))} \left(u'(x) - (\Pi_h u)'(x)\right)^2 dx\right)^{1/2} \le$$

Def $\|\cdot\|_a$

$$\le \left(\max_y (a(y))\right)^{1/2} \cdot \underbrace{\left(\int_0^1 \left(u'(x) - (\Pi_h u)'(x)\right)^2 dx\right)^{1/2}}_{\|u' - (\Pi_h u)'\|_{L^2(0,1)} \le C\cdot\|hu''\|_{L^2}}$$

Th of Chap V

$$\le \left(\max_y (a(y))\right)^{1/2} \cdot C \cdot \|hu''\|_{L^2} \le$$

put back "a"

$$\le C\cdot\left(\max_y (a(y))\right)^{1/2} \left(\int_0^1 \frac{a(x)}{\underbrace{\min\limits_y (a(y))}_{\le 1}} h^2(x) (u''(x))^2 dx\right)^{1/2} \le$$

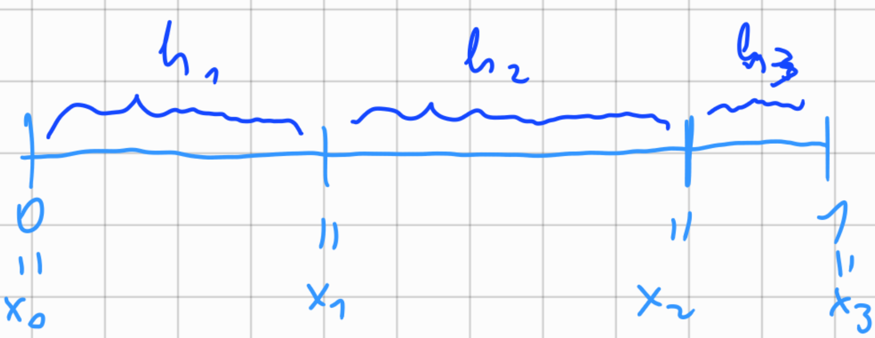
$$\le C \cdot \left(\frac{\max(a(y))}{\min(a(y))}\right)^{1/2} \cdot \underbrace{\left(\int_0^1 a(x)(h u''(x))^2\, dx\right)^{1/2}}_{\|h u''\|_a}$$

$$\le C \cdot \|h u''\|_a \qquad \sum_j \int_{x_{j-1}}^{x_j} h_j$$

3) A posteriori error estimates in the energy norm

error $\le C(h, u_h)$ ← num. sol. by $cG(1)$

<u>not</u> on u



$$h(x) = \begin{cases} h_1 & x_0 < x < x_1 \\ h_2 & x_1 < x < x_2 \\ h_3 & x_2 < x < x_3 \end{cases}$$

uniform $h(x) = h = \frac{1}{m+1}$