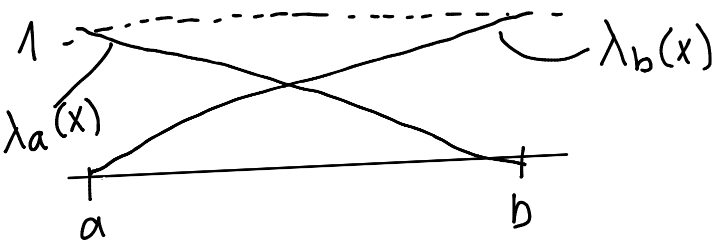
2021-02-05 Exercise Session w.3 Malin Nilsson

6) Prove the following $L_p(a,b)$ error estimates for the interpolation with $p=1$ and $p=2$:

$$\|f-\Pi_1 f\|_{L_p(a,b)} \leq (b-a)^2 \|f''\|_{L_p(a,b)}.$$

Solution. $\Pi_1 f(x) = f(a)\lambda_a(x) + f(b)\lambda_b(x)$



$$\lambda_a(x) = \frac{b-x}{b-a}$$

$$\lambda_b(x) = \frac{x-a}{b-a}$$

$$\lambda_a'(x) = \frac{-1}{b-a} \qquad \lambda_b'(x) = \frac{1}{b-a}$$

useful identities: $\lambda_a + \lambda_b \equiv 1$

$$a\lambda_a(x) + b\lambda_b(x) = x$$

$$(a-x)\lambda_a + (b-x)\lambda_b \equiv 0$$

Assume $f \in C^k(a,b)$ and $f^{(k)}$ is absol. cont.

Taylor expansion: at a point c:

$$f(x) = f(c) + f'(c)(x-c) + \ldots + \frac{f^{(k)}(c)(x-c)^k}{k!} + R_k(x),$$

$$R_k(x) = \int_c^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k\,dt$$

Then for a point $x \in [a,b]$, we have:

$$f(a) = f(x) + f'(x)(a-x) + \int_x^a f''(t)(a-t)\,dt$$

$$f(b) = f(x) + f'(x)(b-x) + \int_x^b f''(t)(b-t)\,dt$$

$$\Rightarrow f(x) - \Pi_1 f(x) = f(x) - \lambda_a(x) f(a) - \lambda_b(x) f(b) =$$

$$f(x) - \left(\lambda_a(x) f(x) + \lambda_a(x) f'(x)(a-x) + \lambda_a(x) \int_x^a f''(t)(a-t)\,dt\right)$$

$$-\left(\lambda_b(x) f(x) + \lambda_b(x) f'(x)(b-x) + \lambda_b(x) \int_x^b f''(t)(b-t)\,dt\right) =$$

$$= f(x)\Big(\underbrace{1 - \underbrace{\lambda_a(x) - \lambda_b(x)}_{-1}}_{0}\Big) - f'(x)\Big(\underbrace{\lambda_a(x)(a-x) + \lambda_b(x)(b-x)}_{0}\Big) -$$

$$- \lambda_a(x) \int_x^a f''(t)(a-t)\,dt - \lambda_b(x) \int_x^b f''(t)(b-t)\,dt =$$

$$= -\lambda_a(x) \int_a^x f''(t)(t-a)\,dt - \lambda_b(x) \int_x^b f''(t)(b-t)\,dt$$

$$\Rightarrow |f(x) - \Pi_1 f(x)| \leq \{\Delta\text{-ineq.}\} \leq$$

$$\leq |\lambda_a(x)| \int_a^x |f''(t)| \underbrace{|t-a|}_{\leq |b-a|} + |\lambda_b(x)| \int_x^b |f''(t)| \underbrace{|b-t|}_{\leq |b-a|} \leq$$

$$\leq \Big(\underbrace{|\lambda_a(x)| + |\lambda_b(x)|}_{=1}\Big) |b-a| \int_a^b |f''(t)|\,dt =$$

$$= |b-a| \int_a^b |f''(t)|\,dt$$

$\underline{p=1}$: $\|f-\Pi_1 f\|_{L_1(a,b)} = \int_a^b |f-\Pi_1 f| \leq$

$$\leq \int_a^b |b-a| \int_a^b |f''(t)|\,dt\,dx =$$

$$= (b-a)^2 \|f''\|_{L_1(a,b)} .$$

$\underline{p=2}$: $\|f-\Pi_1 f\|^2_{L_2(a,b)} = \int_a^b (f-\Pi_1 f)^2 dx \leq$

$$\leq \int_a^b |b-a|^2 \left(\int_a^b 1\cdot|f''(t)|\,dt\right)^2 dx \leq \{C\text{-}S\}$$

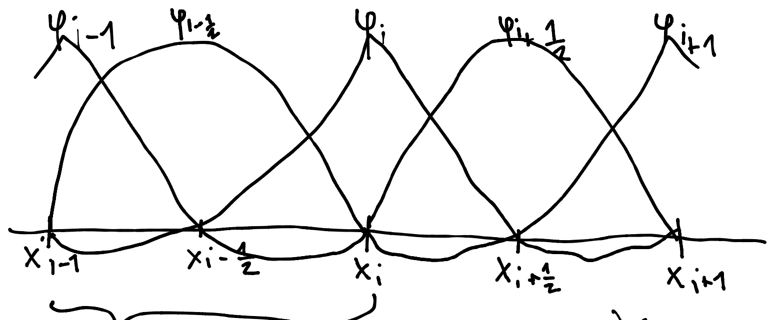
$$\leq \int_a^b |b-a|^2 dx \cdot \int_a^b 1\,dt \cdot \int_a^b |f''(t)|^2 dt =$$

$$= (b-a)^4 \|f''\|^2_{L_2(a,b)} .$$

7) Write down a basis for the set of p.w. quadratic polynomials $W_n^{(2)}$ on (a,b) and plot a sample of the functions.

$\underline{\text{Solution}}$. Let $I=(a,b)$ be partitioned equidistantly $a=x_0<x_{\frac{1}{2}}<x_1<\ldots<x_{n-\frac{1}{2}}<x_n=b$ with $x_i-x_{i-1}=h$

The Lagrange polynomials:



h

$$\varphi_i(x) = \begin{cases} 2(x - x_{i-1})(x - x_{i-\frac{1}{2}})/h^2 & x \in [x_{i-1}, x_i] \\ 2(x_{i+\frac{1}{2}} - x)(x_{i+1} - x)/h^2 & x \in [x_i, x_{i+1}] \\ 0 & \text{else} \end{cases}$$

$$\varphi_{i-\frac{1}{2}}(x) = \begin{cases} 4(x - x_{i-1})(x_i - x)/h^2 & x \in [x_{i-1}, x_i] \\ 0 & \text{else} \end{cases}$$

Use this basis in Ass. 1, ex. 2!

11) Compute the stiffness matrix and load vector for the cG(1) method on a uniform triangulation for

$$\begin{cases} -(a(x)u'(x))' = f(x) & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

with $a(x) = 1 + x$ and $f(x) = \sin(x)$

Solution. Variational formulation: multiply by a test function $v(x) \in H_0^1(0,1)$ and integrate:

$$-\int_0^1 ((1+x)u'(x))' v(x)\,dx = \int_0^1 \sin(x)\, v(x)\,dx$$

$$\text{P.I} \Rightarrow \int_0^1 (1+x)\,u'(x)\,v'(x)\,dx = \int_0^1 \sin(x)\,v(x)\,dx \qquad (*)$$

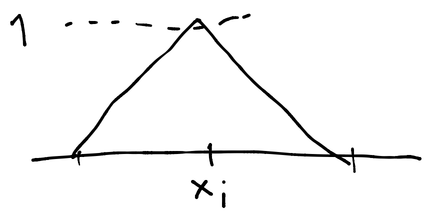
$v(0)=v(1)=0$

Let $\{x_i\}_{i=0}^{m+1}$ be a partition of $[0,1]$ with equidistant step size h.

CG(1) method: $\quad u_h(x) = \sum_{i=1}^{m} \xi_i \varphi_i(x)$

$$v(x) \in \operatorname{span}\left(\{\varphi_j\}_{j=1}^{m}\right)$$

$$\varphi_i(x) = \begin{cases} \frac{x-x_{i-1}}{h} & x\in[x_{i-1},x_i] \\ \frac{x_{i+1}-x}{h} & x\in[x_i,x_{i+1}] \\ 0 & \text{else} \end{cases}$$



$$\varphi_i'(x) = \begin{cases} 1/h & x\in(x_{i-1},x_i) \\ -1/h & x\in(x_i,x_{i+1}) \\ 0 & \text{else} \end{cases}$$

$$(*) \Rightarrow \int_0^1 (1+x) \sum_{i=1}^{m} \xi_i \varphi_i'(x)\varphi_j'(x)\,dx = \int_0^1 \sin(x)\varphi_j(x)\,dx$$

$$\Rightarrow \sum_{i=1}^{m} \xi_i \underbrace{\int_0^1 (1+x)\varphi_i'(x)\varphi_j'(x)\,dx}_{a_{ij}} = \underbrace{\int_0^1 \sin(x)\varphi_j(x)}_{b_j}, \quad j=1,\dots,m$$

<u>Stiffness matrix:</u>

<u>$i=j$</u> $\quad a_{ii} = \int_0^1 (1+x)\left(\varphi_i'(x)\right)^2 =$

$$= \int_{x_{i-1}}^{x_i} (1+x)\frac{1}{h^2} + \int_{x_i}^{x_{i+1}} (1+x)\frac{1}{h^2} = \dots =$$

$$= \frac{2}{h} + \frac{2x_i}{h} = \{x_i = ih\} = \underline{\frac{2}{h} + 2i}$$

$\underline{j=i+1}$ $$a_{i,i+1} = a_{i+1,i} = \int_0^1 (1+x)\varphi_i'(x)\varphi_{i+1}'(x)\,dx$$

$$= -\int_{x_i}^{x_{i+1}} (1+x)\frac{1}{h^2}\,dx = \dots = \underline{-\frac{1}{h} - \frac{2i+1}{2}}$$

$$\Rightarrow A = (a_{ij})_{i,j=1,\dots,m} = \begin{bmatrix} \frac{2}{h}+2 & -\frac{1}{h}-\frac{2+1}{2} & 0 & 0 & \cdots \\ -\frac{1}{h}-\frac{2+1}{2} & \frac{2}{h}+4 & -\frac{1}{h}-\frac{4+1}{2} & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \end{bmatrix}$$

Load vector

$$b_j = \int_0^1 \sin(x)\varphi_j(x)\,dx =$$

$$= \int_{x_{j-1}}^{x_j} \frac{\sin(x)(x-x_{j-1})}{h} + \int_{x_j}^{x_{j+1}} \frac{\sin(x)(x_{j+1}-x)}{h} \overset{PI}{=} \dots$$

$$= \frac{1}{h}\left(2\sin(x_i) - \sin(x_{i-1}) - \sin(x_{i+1})\right)$$

$x_i = ih$

$$b = \frac{1}{h}\begin{bmatrix} 2\sin(h) - \sin(0\cdot h) - \sin(2h) \\ 2\sin(2h) - \sin(h) - \sin(3h) \\ \vdots \end{bmatrix}$$

//.

5.17) Prove an a priori error estimate for the FEM for:

$$\begin{cases} -u''(x) + u'(x) = f(x) & 0<x<1 \\ u(0) = u(1) = 0 \end{cases}$$

Solution. VF: Multiply DE by $v \in H_0^1$ and integrate, P.I. $\Rightarrow$

Find $u \in H_0^1(0,1)$:

$$\underbrace{\int_0^1 u'v' + \int_0^1 u'v}_{:= a(u,v)} = \underbrace{\int_0^1 fv}_{:= F(v)}$$

Note: $a(u,v)$ not symmetric: $\int_0^1 u'v = -\int_0^1 uv'$

$\Rightarrow$ if $u=v$ then $\int_0^1 u'v = 0 \quad \leftarrow (**)$

Let $u_h \in V_0^h$ be the solution to

$$a(u_h, v) = F(v) \quad \forall v \in V_0^h \subset H_0^1$$

$$\|u-u_h\|_E^2 := \int_0^1 (u-u_h)'(u-u_h)' + \int_0^1 (u-u_h)'(u-u_h) =$$

$$= \{(**)\} = \|(u-u_h)'\|^2_{L_2(0,1)}$$

$$\|u-u_h\|_E^2 = a(u-u_h, u-u_h) = \{v \in V_0^h\} =$$

$$= a(u-u_h, u-v+v-u_h) =$$

$$= a(u-u_h, u-v) + a(u-u_h, \underbrace{v-u_h}_{\in V_0^h}) =$$

$$= \left\{ \begin{array}{l} G\perp \text{ (Galerkin orthog.)} \\ a(u-u_h, v) = 0 \quad \forall v \in V_0^h \end{array} \right\}$$

$$= a(u-u_h, u-v) = \int_0^1 (u-u_h)' \Big((u-v)' + (u-v) \Big) \leq$$

$$\leq \{C\text{-}S\} \leq \|(u-u_h)'\|_{L_2(0,1)} \|(u-v)' + (u-v)\|_{L_2(0,1)} \leq$$

$$\leq \{\Delta\text{-ineq.}\} \leq \|(u-u_h)'\|_{L_2} \|(u-v)'\|_{L_2} +$$

$$+ \|(u-u_h)'\|_{L_2} \|u-v\|_{L_2} \leq$$

$$\leq \left\{ \begin{array}{l} u-v \in H_0^1 \\ \text{Poincare} \\ \|u-v\|_{L_2} \leq \|(u-v)'\|_{L_2} \end{array} \right\} \leq$$

$$\leq 2\|(u-u_h)'\|_{L_2} \|(u-v)'\|_{L_2}$$

$$\therefore \|u-u_h\|_E \leq 2\|(u-v)'\|_{L_2} \quad v \in V_0^h$$

Take $v = \Pi_1 u$

$$\|u-u_h\|_E \le 2\|u' - (\Pi_1 u)'\|_{L_2} \le \{\text{thm. } 3.2\} \le$$

$$\le 2Ch\|u''\|_{L_2}.$$

interpolant on this interval size.