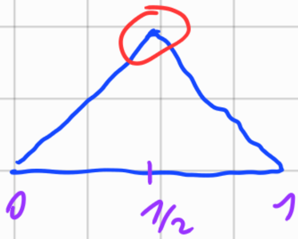
Recall:

- adaptivity (ex. from book Larson + Bengzon, p. 41)

$$\begin{cases} -u''(x) = \hat{\delta}(x) & 0 < x < 1 \\ u(0) = 0, \ u(1) = 0 \end{cases}$$



$\delta(x - 1/2) \leftrightarrow \hat{\delta}(x) = \exp(-c\,|x - 0.5|^2)$, where $c = 100$

-

$$(BVP) \begin{cases} -u''(x) + 4u(x) = 0 & \text{for } 0 < x < 1 \\ u(0) = \alpha, \ u(1) = \beta \end{cases}$$

$\alpha, \beta \neq 0$ given

(VF) Find $u \in V$ s.t. $\int_0^1 u'(x)v'(x)\,dx + 4\int_0^1 u(x)v(x)\,dx \overset{"}{=} 0 \quad \forall v \in V^0$

$V = \{ v \in H^1(0,1) : v(0) = \alpha, v(1) = \beta \}$ trial space

$V^0 = \{ v \in H^1(0,1) : v(0) = 0, v(1) = 0 \} = H_0^1(0,1)$ test space

(FE) Find $u_h \in V_h$ s.t. $\int_0^1 u_h'(x)\chi'(x)\,dx + 4\int_0^1 u_h(x)\chi(x)\,dx \overset{"}{=} 0$

$\forall \chi \in V_h^0$

$V_h = \text{span}(\varphi_0, \varphi_1, \dots, \varphi_m, \varphi_{m+1})$ half hat functions

$V_h^0 = \text{span}(\varphi_1, \dots, \varphi_m)$

$$u_h(x) = \alpha\varphi_0(x) + \sum_{j=1}^{m} \xi_j \varphi_j(x) + \beta\varphi_{m+1}(x)$$

??

Canvas, quiz (today, 23:00), feedback (19:30)

Piazza

Recap & spoiler alert

PDE $\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x,t) & 0<x<1, \quad 0<t<T \\ u(0,t)=0, \ u(1,t)=5 \\ u(\cancel{x},0) = u_0(x) \end{cases}$

x direction

t direction

" $-u_{xx} = f$ "

$u(0)=0, \ u(1)=5$

BVP

" $u_t = f$ "

$u(0) = u_0$

ODE

cG(1)

Euler scheme

C-N

Find a numerical approximation of $u(x,t)$ sol. (PDE)

extend in 2d

(iii) In order to find a linear system for the unknown ζ_j, we insert the above $u_h(x)$ and take $\chi = \varphi_i$, for $i = 1, .., m$, into (FE) problem:

$$\int_0^1 \left(\alpha \varphi_0'(x) + \sum_{j=1}^m \zeta_j \varphi_j'(x) + \beta \varphi_{m+1}'(x)\right) \cdot \varphi_i'(x) dx +$$

$$+ 4 \int_0^1 \left(\alpha \varphi_0(x) + \sum_{j=1}^m \zeta_j \varphi_j(x) + \beta \varphi_{m+1}(x)\right) \cdot \varphi_i(x) dx = 0 \text{ for } i = 1, .., m$$

This gives:

$$\sum_{j=1}^m \zeta_j \underbrace{\int_0^1 \varphi_j'(x) \varphi_i'(x) dx}_{s_{ij}} + 4 \sum_{j=1}^m \zeta_j \underbrace{\int_0^1 \varphi_j(x) \varphi_i(x) dx}_{m_{ij}} =$$

$$= \underbrace{- \alpha \int_0^1 \varphi_0'(x) \varphi_i'(x) dx - \beta \int_0^1 \varphi_{m+1}'(x) \varphi_i'(x) dx - 4 \alpha \int_0^1 \varphi_0(x) \varphi_i(x) dx - 4\beta \int_0^1 \varphi_{m+1}(x) \varphi_i(x) dx}_{b_i} \text{ for } i = 1, .., m.$$

Which is the linear system:

$$S \cdot \zeta + 4 \cdot M \cdot \zeta = b \iff (S + 4M) \zeta = b,$$

where

$S = (s_{ij})_{i,j=1}^{m}$ is the stiffness matrix

$$S = \frac{1}{h}\begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

$M = (m_{ij})_{i,j=1}^{m}$ is the mass matrix

$$M = \frac{h}{6}\begin{pmatrix} 4 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 4 \end{pmatrix}$$ (do at home, see book)

$b = (b_i)_{i=1}^{m}$ is the last vector

$$b = \left(\frac{1}{h} - \frac{2h}{3}\right)\begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \\ \beta \end{pmatrix}$$

α ← BC

← By def. of hat fct.

β ← BC

$\xi = (\xi_j)_{j=1}^{m}$ are the unknown.

Finally, solving the linear system gives ξ and then the FE sol. $u_h(x)$.

Let us a conclude this chapter with a final example.

Consider the following BVP:

$$\begin{cases} -a u''(x) + b u'(x) = r & \text{for } 0 < x < 1 \\ u(0) = 0 \ , \ u'(1) = \beta, \end{cases}$$

where $a > 0$, $b > 0$, r, $\beta \neq 0$ are given.

For ease of presentation, take $a = b = r = 1$ and consider

$$(\text{BVP}) \quad \begin{cases} -u''(x) + u'(x) = 1 \\ u(0) = 0, \ \boxed{u'(1) = \beta} \end{cases}$$

This BC is called inhomogeneous Neumann BC

(provides the flow at $x = 1$)

(i) To get a (VF), we consider the space

$$V = \{ v : [0,1] \to \mathbb{R} : v \in H^1(0,1) \ , \ \underbrace{v(0) = 0}_{\text{hom. Dirichlet BC}} \}$$

Multiply DE with $v \in V$, integrate by parts:

$$\underbrace{-u'(x)\, v(x) \Big|_{x=0}^{1}}_{-u'(1)v(1) + u'(0)v(0)} + \int_0^1 u'(x)\, v'(x)\, dx + \int_0^1 u'(x)\, v(x)\, dx = \int_0^1 1 \cdot v(x)\, dx$$

$\qquad\qquad\qquad\qquad\qquad\qquad \|$

$\qquad \| \qquad\qquad\qquad\qquad 0$ because $v \in V$

$-\beta v(1)$ (because of nonhom. Neumann BC)

Get the variational formulation:

(VF) Find $u \in V$ s.t. $\int_0^1 u'(x)v'(x)\,dx + \int_0^1 u'(x)v(x)\,dx = \int_0^1 v(x)\,dx + \beta v(1)$

$\forall v \in V$

or more compactly

$$(u', v')_{L^2} + (u', v)_{L^2} = (1, v)_{L^2} + \beta v(1).$$

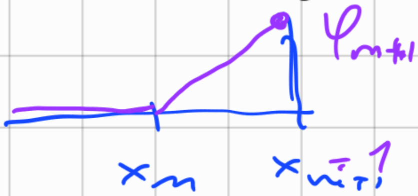
(ii) To find the FE problem, we consider

a partition of $[0,1]$: T_h : $x_0 = 0 < x_1 < x_2 < \dots < x_{m+1} = 1$,

where $h = x_{j+1} - x_j = \frac{1}{m+1}$ and consider FE space

$V_h = \{ v : [0,1] \to \mathbb{R} : v$ is cont. pw linear on T_h, $v(0) = 0 \}$

$= \mathrm{Span}(\varphi_1, \varphi_2, \dots, \varphi_{m+1})$, with hat functions φ_j.



We obtain the FE problem:

(FE) Find $u_h \in V_h$ s.t. $(u_h', \chi') + (u_h', \chi) = (1, \chi) + \beta \chi(1)$

$\forall \chi \in V_h$

(iii) To get the linear system from (FE), we

write $u_h(x) = \sum_{j=1}^{m+1} \xi_j \varphi_j(x)$ and take $\chi = \varphi_i$, $i = 1, \dots, m+1$

into (FE) and obtain:

$$\left(\sum_{j=1}^{m+1} \zeta_j \varphi_j', \varphi_i'\right) + \left(\sum_{j=1}^{m+1} \zeta_j \varphi_j', \varphi_i\right) = (1, \varphi_i) + \beta \varphi_i(1) \quad \text{for } i = 1, \dots, m+1$$

$\Leftrightarrow$

$$\sum_{j=1}^{m+1} \zeta_j \underbrace{(\varphi_j', \varphi_i')}_{s_{ij}} + \sum_{j=1}^{m+1} \zeta_j \underbrace{(\varphi_j', \varphi_i)}_{c_{ij}} = \underbrace{(1, \varphi_i) + \beta \varphi_i(1)}_{b_i} \quad \text{for } i = 1, \dots, m+1$$

$\Leftrightarrow \quad S\zeta + C\zeta = b \Leftrightarrow (S + C) \cdot \zeta = b,$

where $S = (s_{ij})_{i,j=1}^{m+1}$ stiffness matrix

$$S = \frac{1}{h} \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & \boxed{1} \end{pmatrix}$$

$\leftarrow$ "$\varphi_{m+1} = \frac{1}{2}$ hat function"

$C = (c_{ij})_{i,j=1}^{m+1} \rightsquigarrow$ convection matrix

$$C = \frac{1}{2} \begin{pmatrix} 0 & 1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & -1 & 0 \end{pmatrix}$$

$b = (b_i)_{i=1}^{m+1}$ given by $b = \begin{pmatrix} h \\ \vdots \\ h \\ \boxed{\frac{h}{2}} + \boxed{\beta} \end{pmatrix}$

"φ_{m+1} $\frac{1}{2}$ hat"

$\rightarrow$ Neumann BC + def φ_i

Rem: S symmetric :-)

G is not symmetric :-(

Chapter VII: Scalar initial value problem

Goal: Study a particular IVP, define Galerkin schemes for IVP

1) First order linear IVP:

Consider the model problem

$$(\text{IVP}) \begin{cases} \dot{u}(t) + a(t)u(t) = f(t) & \text{for } 0 < t < T \\ u(0) = u_0, \end{cases}$$

f, a are given (cont + bounded)

u_0 given

$$\dot{u}(t) = \frac{d}{dt} u(t)$$

We give a formula for the exact sol.

Th: The sol. to (IVP) is given by the

variation of constants formula (voc)

$$u(t) = u_0 e^{-A(t)} + \int_0^t e^{-(A(t)-A(s))} f(s)\,ds,$$

where $A(t) = \int_0^t a(s)\,ds$ ["easy" if $a(s) \equiv$ constant]

Proof:

- Multiply DE with the integrating factor $e^{A(t)}$:

$$\underbrace{e^{A(t)}\cdot \dot{u}(t) + e^{A(t)}\cdot a(t)u(t)}_{\frac{d}{dt}\left(e^{A(t)}\cdot u(t)\right)} = e^{A(t)}\cdot f(t)$$

$$\frac{d}{dt}\left(e^{A(t)}\cdot u(t)\right) \underset{p}{=} \dot{u}(t)e^{A(t)} + e^{A(t)}\cdot \underset{\underset{a(t)}{\shortparallel}}{\dot{A}(t)}u(t)$$

prod. rule
chain rule

def of $A(t)$

- Next, we integrate the above $\int_0^t$:

$$e^{A(t)}u(t) - \underset{\underset{1}{\shortparallel}}{e^{A(0)}}\underset{\underset{u_0}{\shortparallel}}{u(0)} = \int_0^t e^{A(s)}f(s)ds$$

$$\Leftrightarrow \quad u(t) = e^{-A(t)}u_0 + \int_0^t e^{-(A(t)-A(s))}f(s)ds \quad \leftarrow \text{voc!}$$

$\blacksquare$

We now use the above to investigate the behaviour of sol. to (IVP) for longtime.

Th. Let $u(t)$ be the sol. to (IVP) given by (voc). One has:

(i) If $a(t) \geq \alpha > 0 \ \forall t$, then

$$|u(t)| \leq e^{-\alpha t}|u_0| + \frac{1}{\alpha}\left(1-e^{-\alpha t}\right)\max_{0\leq s\leq t}|f(s)|$$

and the (IVP) is called asymptotically stable

(ii) If $a(t) \geq 0 \ \forall t$, then

$$|u(t)| \leq |u_0| + \int_0^t |f(s)|\, ds$$

and the (IVP) is called stable

Ex: (i) $\dot{u}(t) + \underset{\overset{\shortparallel}{a(t)}}{5}\, u(t) = 1$ has sol. $u(t) \overset{\text{VoC}}{=} (u_0 - \frac{1}{5})e^{-5t} + \frac{1}{5} \xrightarrow[t \to \infty]{} \frac{1}{5}$ $\forall$ initial value u_0

(asym. stable)