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# Chapter 6: FEM for BVP in 1d (summary)

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**Goal**: Present and analyse FEM for several classical BVP.

- Consider the BVP

$$\begin{cases} -(a(x)u'(x))' = f(x) & \text{for} \quad x \in (0,1) \\ u(0) = 0, u(1) = 0, \end{cases}$$

  where for instance $f$ is continuous, $a(x) > 0$, piecewise continuous and bounded on $(0,1)$.

  The above BVP has the following variational formulation (VF)

$$\text{Find} \quad u \in H_0^1 \quad \text{such that} \quad \int_0^1 a(x)u'(x)v'(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x \quad \text{for all} \quad v \in H_0^1$$

  and corresponding FE problem (FE)

$$\text{Find} \quad u_h \in V_h^0 \quad \text{such that} \int_0^1 a(x)u_h'(x)v'(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x \quad \forall v \in V_h^0.$$

  The above is called a cG(1) FEM, for continuous Galerkin (using linear approximation).

  Observing that $V_h^0 \subset H_0^1$, one gets Galerkin orthogonality condition (GO)

$$\int_0^1 a(x)\left(u'(x) - u_h'(x)\right)v'(x)\,\mathrm{d}x = 0 \quad \forall v \in V_h^0$$

  which says that the error of the FE approximation is orthogonal to $V_h^0$ in the energy inner product that we now define.

- For $f, g \in H_0^1$ and $a$ as above, one defines

  the weighted $L_a^2$ inner product

$$(f,g)_a = \int_0^1 f(x)g(x)a(x)\,\mathrm{d}x$$

  the energy inner product

$$(f,g)_E = (f',g')_a$$

  and the corresponding norms

$$\|f\|_a = \sqrt{(f,f)_a} \quad \text{and} \quad \|f\|_E = \sqrt{(f,f)_E}.$$

- A priori error estimate for cG(1): Let $u, u_h$ be the solutions to (VF), resp. (FE). Assume $u'' \in L_a^2(0,1)$. Then, there exists a constant $C > 0$ such that

$$\|u - u_h\|_E \leq C \left\|hu''\right\|_a,$$

  where we recall that $h = h(x)$ is the mesh function of the FE approximation.

- A posteriori error estimate for cG(1): Under technical assumptions on $u$ and $u_h$, one has the following error estimate

$$\|u - u_h\|_E \leq C \left(\int_0^1 \frac{1}{a(x)} h^2(x) R^2(u_h(x))\,\mathrm{d}x\right)^{1/2},$$

  where $R$ denotes the residual $R(u_h) = f(x) + (a(x)u'(x))'$ of the FE approximation to the BVP.

- Adaptivity uses a posteriori error estimates to locally refine or modify the mesh in order to obtain a better approximation $u_h$.

- Let us now derive a FE approximation for the BVP

$$\begin{cases} -u''(x)+4u(x)=0 \quad \text{for} \quad x\in(0,1) \\ u(0)=\alpha \quad \text{and} \quad u(1)=\beta, \end{cases}$$

where $\alpha,\beta\neq 0$ are given real numbers. Such boundary conditions are called non-homogeneous Dirichlet boundary conditions.

The derivation of a numerical approximation for solutions to the above problem is given by

1. Define the trial space $V=\{v\colon[0,1]\to\mathbb{R}\colon\ v\in H^1(0,1), v(0)=\alpha, v(1)=\beta\}$ and the test space $V^0=\{v\colon[0,1]\to\mathbb{R}\colon\ v\in H^1(0,1), v(0)=v(1)=0\}$. Multiply the DE with a test function $v\in V^0$, integrate over the domain $[0,1]$ and get the VF

$$\text{Find} \quad u\in V \quad \text{such that} \quad \int_0^1 u'(x)v'(x)\,\mathrm{d}x+4\int_0^1 u(x)v(x)\,\mathrm{d}x=0 \quad \forall v\in V^0.$$

2. Next, define the finite dimensional spaces
$V_h=\{v\colon[0,1]\to\mathbb{R}\colon\ v \text{ is cont. pw. linear on } T_h \text{ and} v(0)=\alpha, v(1)=\beta\}$ and
$V_h^0=\{v\colon[0,1]\to\mathbb{R}\colon\ v \text{ is cont. pw. linear on } T_h, v(0)=v(1)=0\}$, where as before $T_h$ is a uniform partition with mesh $h=\frac{1}{m+1}$. Observe that $V_h=\operatorname{span}(\varphi_0,\varphi_1,\ldots,\varphi_m,\varphi_{m+1})\subset V$ and $V_h^0=\operatorname{span}(\varphi_1,\ldots,\varphi_m)\subset V^0$ with the hat functions $\varphi_j$.
The FE problem then reads

$$\text{Find} \quad u_h\in V_h \quad \text{such that} \quad \int_0^1 u_h'(x)\chi'(x)\,\mathrm{d}x+4\int_0^1 u_h(x)\chi(x)\,\mathrm{d}x \quad \forall \chi\in V_h^0.$$

3. Choosing $\chi=\varphi_i$, writing $u_h(x)=\sum_{j=0}^{m+1}\zeta_j\varphi_j(x)$ with $\zeta_0=\alpha$ and $\zeta_{m+1}=\beta$ (due to the BC), and inserting everything into the FE problem gives the following linear system of equations

$$(S+4M)\zeta=b,$$

where the $m\times m$ stiffness matrix $S$ has entries $s_{ij}=\int_0^1\varphi_i'(x)\varphi_j'(x)\,\mathrm{d}x$, the $m\times m$ mass matrix $M$ has entries $m_{ij}=\int_0^1\varphi_i(x)\varphi_j(x)\,\mathrm{d}x$, and the $m\times 1$ vector $b$ has entries $b_i=-\alpha(\varphi_0',\varphi_i')_{L^2}-\beta(\varphi_{m+1}',\varphi_i')_{L^2}-4\alpha(\varphi_0,\varphi_i)_{L^2}-4\beta(\varphi_{m+1},\varphi_i')_{L^2}$. Detailed formulas for these entries can be found in the book. Solving this system gives the vector $\zeta$ and in turn the numerical approximation $u_h$.

- Let us finally consider the problem of finding a numerical approximation of solutions to the BVP

$$\begin{cases} -au''(x)+bu'(x)=r \quad \text{for} \quad x\in(0,1) \\ u(0)=0 \quad \text{and} \quad u'(1)=\beta, \end{cases}$$

where $\beta\neq 0$, $a,b>0$, and $r$ are given real numbers. One has a homogeneous Dirichlet boundary conditions for $x=0$ and non-homogeneous Neumann boundary conditions for $x=1$.

For ease of presentation we take $a=b=r=1$ and derive a FE approximation as follows

1. Define the space $V = \{v\colon [0,1] \to \mathbb{R}:\ v \in H^1(0,1), v(0) = 0\}$. Multiply the DE with a test function $v \in V$, integrate over the domain $[0,1]$ and get the VF

$$\text{Find} \quad u \in V \quad \text{such that} \quad (u', v')_{L^2} + (u', v)_{L^2} = \int_0^1 v(x)\,\mathrm{d}x + \beta v(1) \quad \forall v \in V.$$

2. Next, define the finite dimensional space $V_h = \{v\colon [0,1] \to \mathbb{R}:\ v \text{ is cont. pw. linear on } T_h, v(0) = 0\}$, where as before $T_h$ is a uniform partition with mesh $h = \frac{1}{m+1}$.
Observe that $V_h = \operatorname{span}(\varphi_1, \ldots, \varphi_m, \varphi_{m+1}) \subset V$, with the hat functions $\varphi_j$.
The FE problem then reads

$$\text{Find} \quad u_h \in V_h \quad \text{such that} \quad (u_h', \chi')_{L^2} + (u_h', \chi)_{L^2} = \int_0^1 \chi(x)\,\mathrm{d}x + \beta\chi(1) \quad \forall v \in V_h.$$

3. Choosing $\chi = \varphi_i$, writing $u_h(x) = \sum_{j=1}^{m+1} \zeta_j \varphi_j(x)$, observing that $\varphi_{m+1}$ is a half hat function, and inserting everything into the FE problem gives the following linear system of equations

$$(S + C)\zeta = b,$$

where the $(m+1) \times (m+1)$ stiffness matrix $S$ has entries $s_{ij} = \int_0^1 \varphi_i'(x)\varphi_j'(x)\,\mathrm{d}x$, the $(m+1) \times (m+1)$ convection matrix $C$ has entries $c_{ij} = \int_0^1 \varphi_j'(x)\varphi_i(x)\,\mathrm{d}x$, and the $(m+1) \times 1$ vector $b$ has entries $b_i = \int_0^1 \varphi_i(x)\,\mathrm{d}x + \beta\varphi_i(1)$. Detailed formulas for these entries can be found in the book. Solving this system gives the vector $\zeta$ and in turn the numerical approximation $u_h$.

- For indication, and for a uniform partition of $[0,1]$ denoted by $T_h$: $x_0 = 0 < x_1 < x_2 < \ldots < x_m < x_{m+1} = 1$ with element length/mesh denoted by $h$, we summarise the possible choices for the FE spaces:

  1. Dirichlet BC $u(0) = 0, u(1) = 0$: test and trial spaces given by $span(\varphi_1, \ldots, \varphi_m)$.
  2. Dirichlet BC $u(0) = \alpha \neq 0, u(1) = 0$: trial given by $span(\varphi_0, \varphi_1, \ldots, \varphi_m)$ and test by $span(\varphi_1, \ldots, \varphi_m)$.
  3. Dirichlet BC $u(0) = 0, u(1) = \beta \neq 0$: trial given by $span(\varphi_1, \ldots, \varphi_m, \varphi_{m+1})$ and test by $span(\varphi_1, \ldots, \varphi_m)$.
  4. Dirichlet BC $u(0) = \alpha \neq 0, u(1) = \beta \neq 0$: trial given by $span(\varphi_0, \varphi_1, \ldots, \varphi_{m+1})$ and test by $span(\varphi_1, \ldots, \varphi_m)$.
  5. Dirichlet/Neumann BC $u(0) = 0, u'(1) = \beta$ (zero or not): trial given by $span(\varphi_1, \ldots, \varphi_{m+1})$ and test by $span(\varphi_1, \ldots, \varphi_{m+1})$.
  6. Neumann/Dirichlet BC $u'(0) = \alpha$ (zero or not), $u(1) = 0$: trial given by $span(\varphi_0, \ldots, \varphi_m)$ and test by $span(\varphi_0, \ldots, \varphi_m)$.
  7. Dirichlet/Neumann BC $u(0) = \alpha \neq 0, u'(1) = \beta$ (zero or not): trial given by $span(\varphi_0, \ldots, \varphi_{m+1})$ and test by $span(\varphi_0, \ldots, \varphi_{m+1})$.
  8. Neumann/Dirichlet BC $u'(0) = \alpha$ (zero or not), $u(1) = \beta \neq 0$: trial given by $span(\varphi_0, \ldots, \varphi_{m+1})$ and test by $span(\varphi_0, \ldots, \varphi_m)$.
  9. Neumann BC $u'(0) = \alpha, u'(1) = \beta$ (zero or not): trial given by $span(\varphi_0, \ldots, \varphi_{m+1})$ and test by $span(\varphi_0, \ldots, \varphi_{m+1})$.

**Further resources**:

- wikipedia.org
- csc.kth-se
- csc.kth-se