2021-02-10 Exercise session w.4 Malin Nilsson

13) Show that the minimisation problem

$$\left[\begin{array}{l} \text{Find } u_h \in V_h \text{ s.t. } F(u_h) \leq F(v) \ \forall v \in V_h \\ \text{where } F(w) = \frac{1}{2}\int_0^1 a(w')^2\,dx - \int_0^1 fw\,dx \end{array}\right]$$

takes the matrix form

$$\left[\begin{array}{l} \text{Find } \chi = (\chi_j) \in \mathbb{R}^M \text{ that minimizes} \\ \frac{1}{2}\eta^T A\eta - b^T\eta \quad \text{for } \eta \in \mathbb{R}^M \end{array}\right]$$

Solution. $V_h = \text{span}\{\varphi_j, j = 1, \dots, M\}$

Let $w = \sum_{j=1}^{M} \eta_j \varphi_j$

We then have $F(w) = \frac{1}{2}\int_0^1 a\left(\sum_{j=1}^{M}\eta_j\varphi_j'\right)^2 dx -$

$$-\int_0^1 f\sum_{j=1}^{M}\eta_j\varphi_j\,dx = \left\{ \left(\sum_{j=1}^{M}\eta_j\varphi_j'\right)^2 = \sum_{j=1}^{M}\sum_{i=1}^{M}\eta_j\varphi_j'\,\eta_i\varphi_i' \right\} =$$

$$= \frac{1}{2}\sum_{j=1}^{M}\sum_{i=1}^{M}\eta_i\eta_j\int_0^1 a\varphi_j'\varphi_i'\,dx - \sum_{j=1}^{M}\eta_j\int_0^1 f\varphi_j\,dx =$$

$$\left\{ \begin{array}{l} \text{For any vector } x \text{ and matrix } A: \\ [x_1 \dots x_M]\begin{bmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & & \vdots \\ a_{M1} & \cdots & a_{MM} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_M \end{bmatrix} = \dots = \sum_{i=1}^{M}\sum_{j=1}^{M} x_i a_{ij} x_j \end{array} \right\}$$

$= \frac{1}{2}\eta^T A\eta - b^T\eta$ where $\eta = (\eta_i)_1^M$

$b = \left(\int f\varphi_j\, dx\right)_{j=1}^M$ $A = \left(\int a\varphi_j'\varphi_i'\right)_{i,j=1}^M$ //.

14) Consider the Neumann problem

$$\begin{cases} -(a(x)u'(x))' = f(x) & 0<x<1 \\ u(0) = 0, \quad a(1)u'(1) = g_1 \end{cases}$$

VF: Find $U \in V_h$:

$$-\int_0^1 (aU')'v = -[aU'v]_0^1 + \int_0^1 aU'v'dx =$$

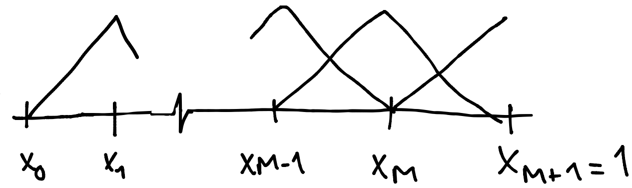
$$-g_1 v(1) + \int_0^1 aU'v'\,dx = \int_0^1 fv\,dx \quad \forall v \in V_h \quad (*)$$

$V_h = \{v \in C(0,1), \text{p.w. lin.}, v(0)=0\}$

$\mathcal{T}_h = \{x_i\}_{i=0}^{M+1}$

* Compute the stiffness matrix, uniform partition, $f = a = g_1 = 1$
* Check if discrete equation corr. to φ_{M+1} (test fcn) at $x=1$ looks like a discrete analogue of the Neumann condition.

Solution. V_h :



$$U = \sum_{i=1}^{M+1} \xi_i \varphi_i$$

$$\Rightarrow (*) = \int_0^1 U'v' - v(1) = \int_0^1 v$$

$$\Rightarrow \sum_{i=1}^{M+1} \xi_i \int \varphi_i' \varphi_j' = \int \varphi_j + \varphi_j(1) \qquad j = 1, \dots, M+1$$

For $i = 1, \dots, M$: $\varphi_i = \frac{1}{h} \begin{cases} x - x_{i-1} & x \in [x_{i-1}, x_i] \\ x_{i+1} - x & x \in [x_i, x_{i+1}] \\ 0 & \text{else} \end{cases} \qquad \varphi_i' = \begin{cases} 1/h \\ -1/h \\ 0 \end{cases}$

$i = M+1$: $\varphi_{M+1} = \frac{1}{h} \begin{cases} x - x_M & x \in [x_M, x_{M+1}] \\ 0 & \text{else} \end{cases} \qquad \varphi_{M+1}' = \begin{cases} 1/h \\ 0 \end{cases}$

Stiffness matrix: $\left(\int_0^1 \varphi_i' \varphi_j' \, dx \right)_{i,j=1}^{M+1}$
$(M+1) \times (M+1)$

For $j = i$, $i = 1, \dots, M$: $\int_0^1 \varphi_i' \varphi_i' = \dots = \frac{2}{h}$

$j = i$, $i = M+1$: $\int_0^1 (\varphi_{M+1}')^2 = \dots = \frac{1}{h}$

$j = i+1$: $\int \varphi_i' \varphi_{i+1}' = \int \varphi_{i+1}' \varphi_i' = \dots = -\frac{1}{h}$

$$A = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & \cdots & & & \\ -1 & 2 & -1 & 0 & \cdots & & \\ & \ddots & \ddots & \ddots & & & \\ & & & -1 & 2 & -1 & 0 \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 1 \end{bmatrix} \begin{bmatrix} \vdots \\ \xi_M \\ \xi_{M+1} \end{bmatrix} = \begin{bmatrix} \vdots \\ b_M \\ b_{M+1} \end{bmatrix}$$

Discrete eq. corr. to φ_{M+1}:

$$-\frac{1}{h}\xi_M + \frac{1}{h}\xi_{M+1} = b_{M+1}$$

$$b_{M+1} = \int_0^1 \varphi_{M+1}\,dx + \varphi_{M+1}(1) =$$

$$= \int_{x_M}^{x_{M+1}} \frac{x - x_M}{h}\,dx + 1 = \frac{h}{2} + 1$$

$$\Rightarrow -\frac{1}{h}\xi_M + \frac{1}{h}\xi_{M+1} = 1 + \frac{h}{2}$$

$$\frac{1}{h}(\xi_{M+1} - \xi_M) = 1 + \frac{h}{2}$$

$$\left\{\begin{matrix} U(x_i) = \xi_i\varphi_i = \xi_i \\ h = x_{i+1} - x_i \end{matrix}\right\}$$

$$\frac{U(x_{M+1}) - U(x_M)}{x_{M+1} - x_M} = 1 + \frac{x_{M+1} - x_M}{2}$$

Let $M \to \infty \Rightarrow U'(1) = 1$ //.

18) Prove an a priori and an a posteriori error estimate for the cG(1) method applied to the BVP

$$(*)\quad \begin{cases} -u'' + bu' + u = f & \text{in } (0,1),\ b \text{ constant} \\ u(0) = u(1) = 0 \end{cases}$$

Solution. Recall

$$\left[\begin{array}{l} \text{A priori: } \|u-u_n\|_E \le \text{smth}(\|u''\|_{L^2}) \\ \text{A posteriori: } \|u-u_n\|_E \le \text{smth}(\|R(u_n)\|_{L^2}) \\ + \text{dependency on } h. \end{array}\right]$$

VF of (x):

$$\int_0^1 u'v' + \int_0^1 bu'v + \int_0^1 uv = \int_0^1 fv \quad \forall v \in H_0^1(0,1)$$

$$cG(1): \int_0^1 u_n'v' + \int_0^1 bu_n'v + \int_0^1 u_n v = \int_0^1 fv \quad \forall v \in V_n$$

$$V_n = \{\text{p.w. lin, } v(0)=v(1)=0\}$$

$$G\perp: \int_0^1 \left((u-u_n)'v' + b(u-u_n)'v + (u-u_n)v\right)dx = 0 \quad \forall v \in V_n$$

or: $a(u-u_n, v) = 0 \quad \forall v \in V_n$

Energy norm: $$\|v\|_E^2 = \int_0^1 \left((v')^2 + \underbrace{bv'v}_{=\frac{b}{2}\frac{d}{dx}(v^2)} + v^2\right) =$$

$$= \int_0^1 \left((v')^2 + v^2\right) + \frac{b}{2}\left[v^2\right]_0^1 \underset{(v=0)}{=} \int_0^1 \left((v')^2 + v^2\right)$$

corr. scalar prod.: $(u,v)_E = \int_0^1 u'v' + uv$

A priori: $e := u - u_n$

$$\|e\|_E^2 = \int \left((e')^2 + be'e + e^2\right)dx =$$

$$= \int_0^1 \left(e'(u-u_h)' + b e'(u-u_h) + e(u-u_h) \right) dx =$$

$$= \{ v \in V_h \} = \int_0^1 \left(e'(u-v)' + b e'(u-v) + e(u-v) \right) dx +$$

$$+ \underbrace{\int_0^1 \left(e'(v-u_h)' + b e'(v-u_h) + e(v-u_h) \right) dx}_{=0,\ v-u_h \in V_h,\ G\perp}$$

$$= \int_0^1 \left(e'(u-v)' + e(u-v) \right) dx + \int_0^1 b e'(u-v)\, dx =$$

$$= (e, u-v)_E + b(e', u-v) \le \{C-S\} \le$$

$$\le \|e\|_E \|u-v\|_E + b \underbrace{\|e'\|_{L^2}}_{\le \|e\|_E} \underbrace{\|u-v\|_{L^2}}_{\le \|u-v\|_E} \le$$

$$\le (1+b) \|e\|_E \|u-v\|_E$$

$$\Rightarrow \|e\|_E \le (1+b) \|u-v\|_E \qquad \forall v \in V_h$$

Let $v = \Pi_1 u$. $\|u-v\|_E^2 = \|u-v\|_{L^2}^2 + \|(u-v)'\|_{L^2}^2$

$$\|u - \Pi_1 u\|_E^2 = \|u - \Pi_1 u\|_{L^2}^2 + \|(u-\Pi_1 u)'\|_{L^2}^2 \le$$

$$\le \left\{ \begin{matrix} \text{thm} \\ 3.2 \end{matrix} \right\} \le (Ch^2 + Dh)^2 \|u''\|_{L^2}^2$$

$$\Rightarrow \|e\|_E \le (1+b)(Ch^2 + Dh) \|u''\|_{L^2(0,1)}$$

A posteriori $\quad R(u_h) = f + u_h'' - bu_h' - u_h$

$$\|e\|_E^2 = \int_0^1 ((e')^2 + be'e + e^2)\,dx =$$

$$= \underbrace{\int_0^1 (u'e' + bu'e + ue)\,dx}_{= \int_0^1 fe\,dx} - \int_0^1 (u_h'e' + bu_h'e + u_he)\,dx$$

$$\left\{\begin{matrix} \Pi_1 e \in V_h \\ u_h \text{ sol.} \\ \text{in } V_h \end{matrix}\right\} = \int_0^1 fe\,dx - \int_0^1 (u_h'e' + bu_h'e + u_he)\,dx +$$

$$+ \underbrace{\int_0^1 (u_h'(\Pi_1 e)' + bu_h'(\Pi_1 e) + u_h(\Pi_1 e))\,dx - \int_0^1 f\Pi_1 e\,dx}_{=0} =$$

$$= \int_0^1 f(e - \Pi_1 e)\,dx - \int_0^1 (\underbrace{u_h'(e - \Pi_1 e)'}_{\substack{\text{want } u_h'' \text{ back} \\ \text{split into subintervals and PI.}}} + bu_h'(e - \Pi_1 e) + u_h(e - \Pi_1 e))\,dx$$

$$= \int_0^1 f(e - \Pi_1 e)\,dx - \int_0^1 (bu_h' + u_h)(e - \Pi_1 e)\,dx + \sum \int_{x_{j-1}}^{x_j} u_h''(e - \Pi_1 e)\,dx =$$

$$= \int_0^1 \underbrace{(f + u_h'' - bu_h' - u_h)}_{R(u_h)}(e - \Pi_1 e)\,dx =$$

$$= \int_0^1 hR(u_h)\frac{e - \Pi_1 e}{h}\,dx \leq \{C\text{-}S\} \leq$$

$$\leq \|hR(u_h)\|_{L^2(0,1)} \left\| \frac{e-\pi_1 e}{h} \right\|_{L^2(0,1)} \leq$$

$$\leq \left\{ \begin{array}{l} \text{thm } 3.2 \quad \left\| \frac{e-\pi_1 e}{h} \right\|_{L^2(I_i)} \leq c\|e'\|_{L^2(I_i)} \\ \sum_i c\|e'\|_{L^2(I_i)} = c\|e'\|_{L^2(I)} \end{array} \right\}$$

$$\leq C\|hR(u_h)\|_{L^2} \|e'\|_{L^2} \leq$$

$$\leq C\|hR(u_h)\|_{L^2} \|e\|_E$$

$$\Rightarrow \|e\|_E \leq Ch\|R(u_h)\|_{L^2(0,1)}. \qquad \%$$