Recall:

- IVP $\begin{cases} \dot{u}(t) + a(t)u(t) = f(t) & 0 < t \leq T \\ u(0) = u_0 \end{cases}$

- (VOC) $u(t) = u_0 e^{-A(t)} + \int_0^t e^{-(A(t)-A(s))} f(s)\, ds,$

  where $A(t) = \int_0^t a(s)\, ds$

- Stability:

  If $a(t) \geq \alpha > 0 \ \forall t \Rightarrow$

  $$|u(t)| \leq e^{-\alpha t} |u_0| + \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right) \max_{0 \leq s \leq t} |f(s)|$$

  IVP asympotically stable

  If $a(t) \geq 0$, then

  $$|u(t)| \leq |u_0| + \int_0^t |f(s)|\, ds$$

  IVP stable

Ex:

- $\begin{cases} \dot{u}(t) + 5u(t) = 1 \\ u(0) = u_0 \end{cases}$

(VOC) $\to u(t) = \left(u_0 - \frac{1}{5}\right) e^{-5t} + \frac{1}{5} \xrightarrow{t \to \infty} \frac{1}{5}$ for any $u_0$!

$\longrightarrow$ asymp. stable.

- $\begin{cases} \dot{u}(t) = \cos(t) \\ u(0) = u_0 \end{cases}$

Sol. $u(t) = \sin(t) + u_0 \xrightarrow{t \to \infty}$ limit depends on $u_0$!

$\longrightarrow$ IVP called stable

- $\begin{cases} \dot{u}(t) - u(t) = 0 \\ u(0) = u_0 \end{cases}$

Sol. $u(t) = u_0 e^{t}$ explodes as $t \to \infty$!

$\to$ IVP unstable.

Proof:

(i) Since $a(s) \geq \alpha > 0 \;\; \forall s \Rightarrow \underbrace{\int_0^t a(s) ds}_{A(t)} \geq \underbrace{\int_0^t \alpha \, ds}_{\alpha t}$

$\Rightarrow A(t) \geq \alpha t \Rightarrow -A(t) \leq -\alpha t \Rightarrow e^{-A(t)} \leq e^{-\alpha t}$

Similarly, for $t > s$, one has $A(t) - A(s) \geq \alpha(t-s)$

$$\rightsquigarrow \quad e^{-(A(t)-A(s))} \leq e^{-\alpha(t-s)}$$

Put this in (VOC) and get:

$$|u(t)| \overset{\Delta}{\underset{\text{above estimates}}{\leq}} e^{-\alpha t}|u_0| + \int_0^t e^{-\alpha(t-s)} \underbrace{|f(s)|}_{\leq \max_{0\leq r\leq t} |f(r)|} ds$$

$$\leq e^{-\alpha t}|u_0| + \max_{0\leq r \leq t}|f(r)| \cdot \int_0^t e^{-\alpha(t-s)} ds$$

$$\leq e^{-\alpha t}|u_0| + \max_{0\leq r\leq t}|f(r)| \cdot \underbrace{\left.\frac{e^{-\alpha(t-s)}}{\alpha}\right|_{s=0}^{t}}_{\frac{1-e^{-\alpha t}}{\alpha}}$$

$$\leq e^{-\alpha t}|u_0| + \max_{0\leq r\leq t}|f(r)| \left(\frac{1-e^{-\alpha t}}{\alpha}\right)$$

(ii) We have $a(s) \geq 0 \;\; \forall s \Rightarrow \underbrace{\int_0^t a(s)\,ds}_{A(t)} \geq 0$

$$\rightsquigarrow \quad -A(t) \leq 0 \rightsquigarrow e^{-A(t)} \leq 1$$

Into (VOC) to get

$$|u(t)| \leq |u_0| + \int_0^t |f(s)|\,ds$$

$\blacksquare$

## 2) Continuous Galerkin methods for IVP:

Prob:

constant (simplicity of presentation)

$$(IVP)\quad \begin{cases} \dot{u}(t) + a\,u(t) = f(t) & 0 < t \leq T \\ u(0) = u_0 . \end{cases}$$

We start with a partition of $[0,T]$:

$0 = t_0 < t_1 < \dots < t_N = T$ for some (LARGE) $N$.

Idea for $cG(1)$ for IVP:

Consider the small intervall $[0,t_1]$.

Test the equation with some $v$ and consider the following approximation $U(t)$:

$$\int_0^{t_1} \left(\dot{U}(t) + a\,U(t)\right) v(t)\,dt = \int_0^{t_1} f(t)\,v(t)\,dt. \qquad (*)$$

To get a $cG(1)$ approximation:

Take $U(t)$ to be of degree 1 (on $[0,t_1]$)

Take $v(t)$ to be of degree $1-1=0$.

Since $\mathcal{U}(t)$ is linear, we can write

$$\mathcal{U}(t) = \underset{\substack{\| \\ u_0}}{\mathcal{U}(0)} \frac{t_1 - t}{t_1} + \underset{\substack{\downarrow \\ b}}{\mathcal{U}(t_1)} \frac{t}{t_1}$$

unknown coordinate

Since $\operatorname{span}(1) = \mathcal{P}^{(0)}(0, t_1) =$ constant polyn.

$\rightsquigarrow$ we can take $v(t) \equiv 1$ (basis).

Next, we put the above in (*):

$$\int_0^{t_1} \left( -\frac{u_0}{t_1} + \frac{\mathcal{U}(t_1)}{t_1} + a\left(u_0 \frac{t_1 - t}{t_1} + \mathcal{U}(t_1) \frac{t}{t_1}\right)\right) \cdot 1 \, dt =$$

$$= \int_0^{t_1} f(t) \cdot 1 \cdot dt$$

We integrate:

$$-u_0 + \mathcal{U}(t_1) + a u_0 \underbrace{\left.\frac{(t_1 - t)^2}{-2t_1}\right|_{t=0}^{t_1}}_{\frac{t_1}{2}} + a\,\mathcal{U}(t_1) \frac{t_1}{2} = \int_0^{t_1} f(t)\,dt$$

This gives:

$$\left(1 + \frac{a t_1}{2}\right) \mathcal{U}(t_1) = \left(1 - \frac{a t_1}{2}\right) u_0 + \underbrace{\int_0^{t_1} f(t)\,dt}_{\text{may use Quadrature formula}}$$

The above gives a formula for $u(t_1)$

and then define the approximation

$u(t)$ given by $cG(1)$ on $[0, t_1]$.

Repeat on $[t_1, t_2]$, etc.

Start at $u(t_1)$

**Rem:**
- $cG(1)$ applied to linear problem (with trapezoidal rule) is C-N
- See book for $cG(q)$ for $q \in \mathbb{N}, q > 1$. p. 154 (if interested)

## 3) Discontinuous Galerkin methods for IVP

**Idea:** Use discontinuous approximation:


(+) Treats various types of elements ( ) and irregular mesh

(+) Hyperbolic problems, shocks

The discontinuous Galerkin method of degree $q$, $dG(q)$ for IVP, reads

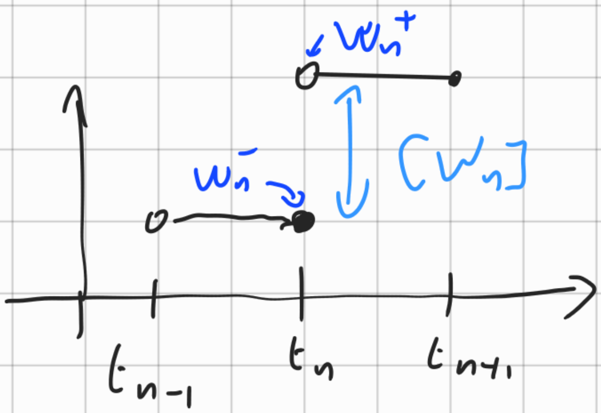
For $n = 1, 2, \dots, N$, find $\mathcal{U}(t) \in \mathcal{P}^{(q)}(t_{n-1}, t_n)$ s.t.

$$\int_{t_{n-1}}^{t_n} \left(\dot{\mathcal{U}}(t) + a\,\mathcal{U}(t)\right) v(t)\,dt + \mathcal{U}_{n-1}^{+} v_{n-1}^{+} =$$

$$= \int_{t_{n-1}}^{t_n} f(t)\, v(t)\,dt + \mathcal{U}_{n-1}^{-} v_{n-1}^{+} \quad \forall v \in \mathcal{P}^{(q)}(t_{n-1}, t_n)$$

where $\mathcal{P}^{(q)}(t_{n-1}, t_n) \rightsquigarrow$ polyn. of degree $q$ in $(t_{n-1}, t_n]$

and $w_n^{\pm} = \lim_{\varepsilon \to 0^+} w(t_n \pm \varepsilon)$




Jump

$$[w_n] = w_n^+ - w_n^-$$

Rem: The term $\mathcal{U}_{n-1}^{+} v_{n-1}^{+} - \mathcal{U}_{n-1}^{-} v_{n-1}^{+} =$

$$= \left(\mathcal{U}_{n-1}^{+} - \mathcal{U}_{n-1}^{-}\right) v_{n-1}^{+} = \underbrace{[\mathcal{U}_{n-1}]}_{\text{jump}} \cdot v_{n-1}^{+}$$

$\rightsquigarrow$ this term "average the jump" and tries to connect the value of the approximation on 2 intervals.

<u>Ex:</u> $dG(0)$

$q=0 \rightsquigarrow$ test function $v \in \mathcal{P}^{(0)}(t_{n-1}, t_n) = \text{span}(1)$

constant

$\rightsquigarrow$ approximation $U(t) = \text{CONSTANT} =$

$= U_n = U^+_{n-1} = U^-_n$ on $(t_{n-1}, t_n]$.

Next, we insert the above into the definition of $dG(0)$ and get:

$$\int_{t_{n-1}}^{t_n} \Big(\underset{\substack{\| \\ 0 \;\leftarrow\; \text{constant}}}{\dot U(t)} + a \underset{\substack{\| \\ U_n}}{U(t)}\Big) \underset{\substack{\| \\ 1 \\ \text{(see above)}}}{v(t)}\, dt + \underset{\substack{\| \\ U_n}}{U^+_{n-1}}\, \underset{\substack{\| \\ 1}}{v^+_{n-1}} =$$

$$= \int_{t_{n-1}}^{t_n} f(t) \underset{\substack{\| \\ 1}}{v(t)}\, dt + \underset{\substack{\| \\ U_{n-1}}}{U^-_{n-1}}\, \underset{\substack{\| \\ 1}}{v^+_{n-1}}$$

$$\int_{t_{n-1}}^{t_n} a\underbrace{U_n}_{\text{constant}}\,dt + U_n = \int_{t_{n-1}}^{t_n} f(t)\,dt + U_{n-1}$$

$$ak_n \cdot U_n + U_n = \int_{t_{n-1}}^{t_n} f(t)\,dt + U_{n-1}$$

$$\Leftrightarrow \quad U_n = U_{n-1} - ak_n U_n + \int_{t_{n-1}}^{t_n} f(t)\,dt$$

$\rightsquigarrow$ this gives $U_n$ and thus $U(t) = U_n$ on $(t_{n-1}, t_n]$ and then repeat $(t_n, t_{n+1}]$, etc

__Rem:__ • If $\int_{t_{n-1}}^{t_n} f(t)\,dt \approx f(t_n)\cdot k_n$ then

dG(0) = implicit/backward Euler scheme.

## 4) A posteriori error estimates for IVP:

__Prob:__ (IVP) $\begin{cases} \dot{u}(t) + a(t)\,u(t) = f(t) & 0 < t \le T \\ u(0) = u_0 \end{cases}$

Th) Denote by $U(t)$ the approximation of IVP given by cG(1) on partition $\mathcal{T}_k$: with $k(t) = k_n = t_n - t_{n-1}$ for $t \in [t_{n-1}, t_n]$,

Then,

$$|u(t) - U(t)| \leq S(T) \max_{t \in [0,T]} \left| \boxed{k(t)^2} \cdot \frac{d}{dk}\left(a(t)U(t) - f(t)\right) \right|,$$

where $S$ is a stability function found book eq. (6.4.7)

Rem: $S(T)$ can be bounded
If one can compute and bound $\frac{d}{dk}(\ldots)$
and one take constant stepsize $k(t) = k$
$\rightsquigarrow$ $|\text{error } cG(1)| \leq C \cdot k^{\boxed{2}}$

For discontinuous Galerkin, we get the following result:

Th: Let $\mathcal{U}(t)$ denotes the approximation given by $dG(0)$. Then,

$$|u(t) - \mathcal{U}(t)| \leq S'(T) \cdot \max_{t \in [0,T]} |k(t) \cdot R(\mathcal{U}(t))|,$$

where $S'$ is again the above stability function, and $R$ is a residual, see book.

Rem: $S'$ can be bounded and $R$ also (work)

$\rightsquigarrow$ $|$error of $dG(0)| \leq C \cdot \boxed{k^1}$

(in case of uniform partition)

5) A priori error estimates for $dG(0)$: