<u>Recall:</u>

- IVP $\begin{cases} \dot{u}(t) + a u(t) = f(t) \\ u(0) = u_0 \end{cases}$

- VOC + stability for $u(t)$

- $cG(1)$ $\rightsquigarrow$ $U(t) = u_0 \frac{t_1 - t}{t_1} + U(t_1)\frac{t}{t_1}$ for $t \in [0, t_1]$

  where $\left(1 + \frac{a t_1}{2}\right) U(t_1) = \left(1 - \frac{a t_1}{2}\right) u_0 + \int_0^{t_1} f(t)\,dt$

  + appropriate quadrature formula gives C-N

- $dG(0)$ $\rightsquigarrow$ $U(t) = U_n = U_{n-1} - k a U_n + \int_{t_{n-1}}^{t_n} f(t)\,dt$ for $t \in (t_{n-1}, t_n]$

  + appropriate quadrature formula gives back. Euler

5) A priori error estimates for dG(0):

Prob: $\begin{cases} \dot{u}(t) + a\,u(t) = f(t) & 0 < t \leq T \\ u(0) = u_0 \end{cases}$

Th: Denote by $U(t)$ the approximation of $u(t)$ given by dG(0), denote by k_n the length of the small interval $[t_{n-1}, t_n]$.

If $k_n |a| \leq 1/2$ for $n = 1, 2, 3, \dots$, then

$$|u(T) - U(T)| \leq \frac{e}{4}\left(e^{2|a|T} - 1\right) \max_{1 \leq n \leq N} \left(\boxed{k_n} \max_{t \in [t_{n-1}, t_n]} |\dot{u}(t)|\right)$$

Error $\sim \mathcal{O}(k^1)$ when one has uniform $k_n = k$.

Chapter VIII: The heat equation in 1d

Goal: Discuss stability of exact sol.
Present a numerical method for heat eq.

1) Heat equations in 1d:

The following is a simple model to describe heat flow on a thin wire / metal rod of length 1:

(Newton' law cooling)

$$(H)\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x,t) & 0<x<1,\ 0<t\leq T & \text{(DE)} \\ u(0,t)=0,\ u_x(1,t)=0 & 0<t\leq T & \text{(BC)} \\ u(x,0) = u_0(x) & 0<x<1 & \text{(IC)} \end{cases}$$

Here; $u(x,t)$ .... unknown $\rightsquigarrow$ temperature at position x and time t

$f(x,t)$ .... given $\rightsquigarrow$ heat source / sink

$u_0(x)$ .... given $\rightsquigarrow$ initial temperature profile

$u(0,t)=0$ .... homog. Dirichlet BC $\rightsquigarrow$ fixed temp. 0 at $x=0$

$u_x(1,t)=0$ ... homog. Neumann BC $\rightsquigarrow$ no heat flux at $x=1$.

<u>Th</u>: The sol. to (H) satisfies the following stability estimates:

(i) $$\|u(\cdot,t)\|_{L^2(0,1)} \leq \|u_0\|_{L^2(0,1)} + \int_0^t \|f(\cdot,s)\|_{L^2(0,1)}\,ds$$

(remember last chapter)

(ii) $$\|u_x(\cdot,t)\|_{L^2(0,1)} \leq \|u_0'\|_{L^2(0,1)} + \int_0^t \|f(\cdot,s)\|_{L^2(0,1)}\,ds$$

(iii) When $f \equiv 0$, one has

$$\|u(\cdot,t)\|_{L^2(0,1)} \leq \|u_0\|_{L^2(0,1)} \cdot e^{-2t}$$

(typical for parabolic)

Rem: • Notation: $\|u(\cdot,t)\|^2_{L^2(0,1)} = \int_0^1 |u(x,t)|^2 dx$

• (iii) says that the temperature goes to zero in L^2-norm as $t \to \infty$.

Proof:

(i) Multiply (DE) with u and integrate $\int_0^1 dx$:

$$\underbrace{\int_0^1 u_t(x,t)\, u(x,t)\, dx}_{} - \underbrace{\int_0^1 u_{xx}(x,t)\, u(x,t)\, dx}_{\text{by part}} = \underbrace{\int_0^1 f(x,t)\, u(x,t)\, dx}_{}$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left((u(x,t))^2\right) \underset{\text{Chain rule}}{=} \frac{2}{2} u(x,t) \frac{\partial}{\partial t} u(x,t)$$

Integrate by part:

$$\underbrace{\frac{1}{2}\frac{d}{dt}\int_0^1 (u(x,t))^2 dx}_{\|u(\cdot,t)\|^2_{L^2(0,1)}} - \underbrace{u_x(x,t)\,u(x,t)}_{= 0 \text{ by BC}}\Big|_{x=0}^{1} + \underbrace{\int_0^1 u_x(x,t)\,u_x(x,t)\, dx}_{\|u_x(\cdot,t)\|^2_{L^2(0,1)}} =$$

Hence, we get:

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2(0,1)} + \underbrace{\|u_x(\cdot,t)\|^2_{L^2(0,1)}}_{\geq 0} = \underbrace{\int_0^1 f(x,t)u(x,t)\,dx}_{C\text{-}S} \quad (*)$$

Using C-S, we get:

$$\underbrace{\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2(0,1)}}_{\text{Chain rule}} \leq \|f(\cdot,t)\|_{L^2(0,1)} \cdot \|u(\cdot,t)\|_{L^2(0,1)}$$

$$\frac{2}{2}\|u(\cdot,t)\|_{L^2}\cdot\frac{d}{dt}\|u(\cdot,t)\|_{L^2(0,1)}$$

This gives

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^2(0,1)} \leq \|f(\cdot,t)\|_{L^2(0,1)}$$

Finally, integrate in time: $\int_0^t \ldots ds$

$$\|u(\cdot,t)\|_{L^2(0,1)} - \underbrace{\|u(\cdot,0)\|_{L^2}}_{\|u_0\|_{L^2(0,1)}} \leq \int_0^t \|f(\cdot,s)\|_{L^2(0,1)}\,ds$$

:-)

(ii) $\rightsquigarrow$ book if interested

(iii) We assume $f\equiv 0$ and consider $(*)$:

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2(0,1)} + \|u_x(\cdot,t)\|^2_{L^2(0,1)} = 0$$

Remembering Poincaré (1d) $\|u(\cdot,t)\|^2_{L^2(0,1)} \leq \frac{1}{2}\|u_x(\cdot,t)\|^2_{L^2(0,1)}$

we get: [need $v(0)=0$ and not $v \in H^1_0$]

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2(0,1)} + 2\|u(\cdot,t)\|^2_{L^2(0,1)} \leq 0$$

Integrating factor $\cdot 2e^{4t}$ to get:

$$\underbrace{\frac{d}{dt}\left(\|u(\cdot,t)\|^2_{L^2(0,1)}\right)e^{4t} + 4e^{4t}\|u(\cdot,t)\|^2_{L^2(0,1)}}_{\frac{d}{dt}\left(e^{4t}\|u(\cdot,t)\|^2_{L^2(0,1)}\right)} \leq 0$$

$$\frac{d}{dt}\left(e^{4t}\|u(\cdot,t)\|^2_{L^2(0,1)}\right) \underset{\text{prod rule + chain}}{=} 4e^{4t}\cdot\|u\|^2 + e^{4t}\frac{d}{dt}\|u\|^2$$

Finally, we integrate in time $\int_0^t \dots ds$:

$$e^{4t}\|u(\cdot,t)\|^2_{L^2(0,1)} - \|u_0\|^2_{L^2(0,1)} \leq 0$$

$$\Longrightarrow \quad \|u(\cdot,t)\|_{L^2(0,1)} \leq e^{-2t}\|u_0\|_{L^2(0,1)}$$

:-)

Th: (Energy estimate)

Consider (H) with $f \equiv 0$ and let $\varepsilon > 0$. Then,

$$\int_{\boxed{\varepsilon}}^{t} \|u_t(\cdot,s)\|_{L^2(0,1)}\, ds \le \frac{1}{2}\sqrt{\underset{\uparrow \text{log}}{\ln}\left(\frac{t}{\boxed{\varepsilon}}\right)}\, \|u_0\|_{L^2(0,1)}$$

for all $t \in [\varepsilon, T]$.

Proof:

- Multiply DE $-t\,u_{xx}$ and $\int_0^1 \ldots dx$ to get:

$$\underbrace{-t\int_0^1 u_t\, u_{xx}\, dx}_{\text{by parts}} + \underbrace{t\int_0^1 u_{xx}\cdot u_{xx}\, dx}_{= \|u_{xx}\|^2_{L^2}} = 0$$

$$\underbrace{u_t u_x \Big|_{x=0}^{1}}_{\substack{\| \\ 0 \text{ by BC } (u(0,t)=0,\ u_x(1,t)=0) \\ \Rightarrow u_t(0,t)=0}} - \int_0^1 \underbrace{u_{tx}\, u_x}_{\frac{1}{2}\frac{d}{dt} u_x^2} dx$$

We obtain:

$$\frac{t}{2}\frac{d}{dt}\|u_x\|^2_{L^2} + t\,\|u_{xx}\|^2_{L^2} = 0 \qquad (+)$$

• Next, observe that:

$$t\frac{d}{dt}\|u_x\|_{L^2}^2 = \underbrace{\frac{d}{dt}\left(t\|u_x\|_{L^2}^2\right)}_{\text{product rule}, \; t\frac{d}{dt}\|u_x\|_{L^2}^2 + 1\cdot\|u_x\|_{L^2}^2} - \|u_x\|_{L^2}^2$$

and thus (+) reads

$$\frac{1}{2}\frac{d}{dt}\left(t\|u_x\|_{L^2}^2\right) - \frac{1}{2}\|u_x\|_{L^2}^2 + t\|u_{xx}\|_{L^2}^2 = 0$$

$$\Leftrightarrow \frac{d}{dt}\left(t\|u_x\|_{L^2}^2\right) + 2t\|u_{xx}\|_{L^2}^2 = \|u_x\|_{L^2}^2$$

• Integrate in time $\int_0^t \dots ds$ and get:

$$t\|u_x\|_{L^2}^2 - 0 + 2\int_0^t s\|u_{xx}\|_{L^2}^2\, ds = \int_0^t \|u_x\|_{L^2}^2\, ds \underset{(*)}{\leq} \frac{1}{2}\|u_0\|_{L^2}^2$$

Show (*):

From previous proof: $\frac{1}{2}\frac{d}{ds}\|u(\cdot,s)\|_{L^2}^2 + \|u_x(\cdot,s)\|_{L^2}^2 = 0$

$\underset{\sim}{\text{integrate}}$ $\int_0^t \left(\frac{1}{2}\frac{d}{ds}\|u(\cdot,s)\|_{L^2}^2 + \|u_x(\cdot,s)\|_{L^2}^2\right) ds = 0$

$$\Leftrightarrow \frac{1}{2}\|u(\cdot,t)\|_{L^2}^2 - \frac{1}{2}\|u_0\|_{L^2}^2 + \int_0^t \|u_x(\cdot,s)\|_{L^2}^2\, ds = 0$$

$$\Leftrightarrow \int_0^t \|u_x(\cdot,s)\|_{L^2}^2\,ds = \frac{1}{2}\|u_0\|_{L^2}^2 - \underbrace{\frac{1}{2}\|u(\cdot,t)\|_{L^2}^2}_{\geq 0} \leq \frac{1}{2}\|u_0\|_{L^2}^2$$

This $(*)$

- The above provides:

$$\underbrace{t\|u_x(\cdot,t)\|_{L^2}^2}_{\geq 0} + 2\underbrace{\int_0^t s\|u_{xx}(\cdot,s)\|_{L^2}^2\,ds}_{\geq 0} \leq \frac{1}{2}\|u_0\|_{L^2}^2$$

Which implies

$$2\int_0^t s\|u_{xx}(\cdot,s)\|_{L^2}^2\,ds \leq \frac{1}{2}\|u_0\|_{L^2}^2$$

$$\Leftrightarrow \left(\int_0^t s\|u_{xx}(\cdot,s)\|_{L^2}^2\,ds\right)^{1/2} \leq \frac{1}{2}\|u_0\|_{L^2} \quad X$$

- Finally, we use again DE $u_t - u_{xx} = 0 \Leftrightarrow$

$u_t = u_{xx} \Leftrightarrow \|u_t\|_{L^2(0,1)} = \|u_{xx}\|_{L^2(0,1)}$ and integrate in time:

$$\int_\varepsilon^t \|u_t(\cdot,s)\|_{L^2(0,1)}\,ds = \int_\varepsilon^t \|u_{xx}(\cdot,s)\|_{L^2(0,1)}\sqrt{s}\,\frac{1}{\sqrt{s}}\,ds \overset{C-S}{\leq}$$

$$\leq \left(\int_\varepsilon^t \underbrace{\frac{1}{(\sqrt{s})^2}}_{\text{exact}}\,ds\right)^{1/2} \cdot \left(\underline{\int_\varepsilon^t \|u_{xx}(\cdot,s)\|_{L^2}^2 \cdot s\,ds}\right)^{1/2}$$

(X) $\leq \frac{1}{2} \sqrt{\ln\left(\frac{t}{\varepsilon}\right)} \cdot \|u_0\|_{L^2}^2$

$\blacksquare$