Chapter 7: Scalar initial value problems (summary)

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Goal: Study exact solutions to particular IVPs and present Galerkin discretisations of these IVPs.

- Let $f, a\colon (0,T] \to \mathbb{R}$ be continuous and bounded for instance. Let $u_0 \in \mathbb{R}$. Consider the first order linear DE
$$\begin{cases} \dot{u}(t) + a(t)u(t) = f(t) \quad \text{for} \quad t \in (0,T] \\ u(0) = u_0. \end{cases}$$
The exact solution to the above IVP is given by the variation of constants formula (voc)
$$u(t) = u_0 \mathrm{e}^{-A(t)} + \int_0^t \mathrm{e}^{-(A(t)-A(s))} f(s)\,\mathrm{d}s,$$
where $A(t) = \displaystyle\int_0^t a(s)\,\mathrm{d}s$.
If $a(t) \geq 0$ for all $t \in (0,T]$, we have the following stability estimate
$$|u(t)| \leq |u_0| + \int_0^t |f(s)|\,\mathrm{d}s$$
and the above IVP is called stable.
If $a(t) \geq \alpha > 0$ for all $t \in (0,T]$, we have the following stability estimate
$$|u(t)| \leq \mathrm{e}^{-\alpha t}|u_0| + \frac{1}{\alpha}(1-\mathrm{e}^{-\alpha t}) \max_{0\leq s\leq T} |f(s)|$$
and the above IVP is called asymptotically stable.
If $a(t) < 0$, the above IVP is called unstable.

- We define the continuous Galerkin scheme cG(1) for the following IVP
$$\begin{cases} \dot{u}(t) + au(t) = f(t) \quad \text{for} \quad t \in (0,T] \\ u(0) = u_0. \end{cases}$$
Consider first a partition of the interval $[0,T]$ given by $0 = t_0 < t_1 < \ldots < t_N = T$ and the following equation on the small interval $[0,t_1]$:
Find $U(t) \in \mathscr{P}^{(1)}(0,t_1)$ s.t.
$$\int_0^{t_1} \big(\dot{U}(t) + aU(t)\big)\, v(t)\,\mathrm{d}s = \int_0^{t_1} f(t)v(t)\,\mathrm{d}s \quad \text{for all} \quad v \in \mathscr{P}^{(0)}(0,t_1). \tag{1}$$
By definition of these polynomial spaces, one can take $v(t) = 1$ (constant) and
$$U(t) = u_0 \frac{t_1 - t}{t_1} + U(t_1)\frac{t}{t_1} \quad \text{(linear)}.$$
Insert this in the equation (1) gives the following formula for the unknown $U(t_1)$:
$$(1 + \frac{at_1}{2})U(t_1) = (1 - \frac{at_1}{2})u_0 + \int_0^{t_1} f(t)\,\mathrm{d}t.$$
Inserting the above in the definition of $U(t)$ provides the approximation by cG(1) on the first interval $[0,t_1]$. Repeat this procedure in the next interval.

- The following generalisation of cG(1) is called a discontinuous Galerkin scheme. We briefly illustrate this procedure for a positive integer q:

  For $n = 1, 2, \dots, N$, find $U(t) \in \mathscr{P}^{(q)}(t_{n-1}, t_n)$ such that

  $$\int_{t_{n-1}}^{t_n} \big(\dot U(t) + aU(t)\big)\, v(t)\,\mathrm{d}t + U_{n-1}^+ v_{n-1}^+ = \int_{t_{n-1}}^{t_n} f(t) v(t), \mathrm{d}t + U_{n-1}^- v_{n-1}^+ \quad \text{for all} \quad v \in \mathscr{P}^{(q)}(t_{n-1}, t_n).$$

  For $q = 0$, one gets the dG(0) scheme (defined on the interval $[t_{n-1}, t_n]$):

  $$U(t) = U_n = U_{n-1} - k_n a U_n + \int_{t_{n-1}}^{t_n} f(t)\,\mathrm{d}t,$$

  where k_n denotes the length of the considered interval.

- We finally state some error estimates for continuous and discontinuous Galerkin schemes for IVP.

  First, consider the initial value problem

  $$\begin{cases} \dot u(t) + a(t)u(t) = f(t) & \text{for} \quad t \in (0, T] \\ u(0) = u_0. \end{cases}$$

  A posteriori error estimates for cG(1): Let $U(t)$ denote the cG(1) approximation of the exact solution $u(t)$ of the above problem on a partition with time step-size $k(t) = k_n = t_n - t_{n-1}$ for $t \in (t_{n-1}, t_n]$. Then, one has

  $$|u(t) - U(t)| \le S(T) \max_{t \in [0,T]} |k(t)^2 \frac{\mathrm{d}}{\mathrm{d}t}(a(t)U(t) - f(t))|,$$

  where S is some stability function, see the book for details.

  A posteriori error estimates for dG(0): Let $U(t)$ denote the dG(0) approximation of the exact solution $u(t)$ of the above problem on a partition with time step-size $k(t) = k_n = t_n - t_{n-1}$ for $t \in (t_{n-1}, t_n]$. Then, one has

  $$|u(t) - U(t)| \le S(T) \max_{t \in [0,T]} |k(t) R(U(t))|,$$

  where R is some residual function, see the book for details.

  Next, consider

  $$\begin{cases} \dot u(t) + au(t) = f(t) & \text{for} \quad t \in (0, T] \\ u(0) = u_0. \end{cases}$$

  A priori error estimates for dG(0): Let $U(t)$ denote the dG(0) approximation of the exact solution $u(t)$ of the above problem on a partition with time step-size $k(t) = k_n = t_n - t_{n-1}$ for $t \in (t_{n-1}, t_n]$. Under some assumptions, one has

  $$|u(T) - U(T)| \le \frac{\mathrm{e}}{4}(\mathrm{e}^{2|a|T} - 1) \max_{1 \le n \le N} (k_n \max_{t \in (t_{n-1}, t_n]} |\dot u(t)|).$$

  Further results for the case $a(t) \ge 0$ or for cG(1) can be found in the book.

Further resources:

- whitman.edut'
- britannica.com