Recall:

- Heat eq.
$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x,t) & 0<x<1 \\ u(0,t) = 0, \; u_x(1,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

- Stability estimates, f. ex:
$$\|u(\cdot,t)\|_{L^2(0,1)} \leq \|u_0\|_{L^2(0,1)} + \int_0^t \|f(\cdot,s)\|_{L^2(0,1)}\, ds$$

$$\left[\quad \begin{array}{l} \dot{u}(t) + a u(t) = f(t) \rightsquigarrow |u(t)| \leq |u_0| + \int_0^t |f(s)|\, ds \\ u(0) = u_0 \end{array} \quad \right]$$

2) Discretisation of the heat equation:

Consider the PDE:

$$(H)\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x,t) & 0<x<1,\ 0<t<T \\ u(0,t)=0,\ u(1,t)=0 & 0<t<T \\ u(x,0) = u_0(x) & 0<x<1 \end{cases}$$

(i) In order to obtain a variational formulation in space of (H), we consider the space $V^0 = H_0^1(0,1)$ and test the equation with a test function $v \in V^0$ (+ integration by part):

$$\int_0^1 u_t(x,t)\,v(x)\,dx - \underbrace{\int_0^1 u_{xx}(x,t)\,v(x)\,dx}_{\underbrace{u_x(x,t)\,v(x)\Big|_0^1}_{=0 \text{ since } v\in V^0 = H_0^1(0,1)} - \int_0^1 u_x(x,t)\,v'(x)\,dx} = \int_0^1 f(x,t)\,v(x)\,dx$$

We thus obtain: For $0<t<T$, find

$$(VF)\quad u(\cdot,t) \in V^0 \ \text{ s.t. } \int_0^1 u_t(x,t)\,v(x)\,dx + \int_0^1 u_x(x,t)\,v'(x)\,dx = \int_0^1 f(x,t)\,v(x)\,dx \quad \forall v \in V^0$$

Compact form:

$$(u_t(\cdot,t),v)_{L^2(0,1)} + (u_x(\cdot,t),v')_{L^2} = (f(\cdot,t),v)_{L^2}$$

(ii) To get a finite element problem, consider

a partition of $[0,1]$: T_h: $0 = x_0 < x_1 < x_2 < \dots < x_{m+1} = 1$

with $h = \frac{1}{m+1} = x_j - x_{j-1}$ is the mesh of the FEM.

Consider the space

$$V_h^0 = \{v:[0,1]\to\mathbb{R},\ v \text{ continuous pw linear on } T_h,\ v(0)=0,\ v(1)=0\}$$

Then, we get the FE problem:

For $0 < t \leq T$, find $\mathcal{U}(\cdot,t) \in V_h^0$ s.t.

(FE) $$(\mathcal{U}_t(\cdot,t),\chi)_{L^2} + (\mathcal{U}_x(\cdot,t),\chi')_{L^2} = (f(\cdot,t),\chi) \quad \forall \chi \in V_h^0$$

$$\mathcal{U}(x,0) = \pi_h u_0(x)$$

V_h^0

cont. pw linear interpolant of u_0

⚠ $\pi_h u_0 \in V_h^0$.

(iii) Next, we need to find a system of ODE.

To do this, we make the following observations:

$V_h^0 := \text{span}(\varphi_1, \varphi_2, \dots, \dots, \varphi_m)$, where φ_i are hat functions

Since $\mathcal{U}(\cdot, \underline{t}) \in V_h^0 \Rightarrow \mathcal{U}(x,t) = \sum_{j=1}^{m} \boxed{\xi_j(t)} \varphi_j(x)$

unknown coeff. (time-dependent)

Take $\chi(x) = \varphi_i(x)$ for $i = 1, 2, \dots, m$.

We put the above in (FE) and obtain:

$$\left(\sum_{j=1}^{m} \dot{\xi}_j(t) \varphi_j, \varphi_i \right)_{L^2} + \left(\sum_{j=1}^{m} \xi_j(t) \varphi_j', \varphi_i' \right)_{L^2} = (f(\cdot, t), \varphi_i)_{L^2}$$

for $i = 1, 2, \dots, m$.

For initial condition, we get

$$\mathcal{U}(x,0) = \Pi_h u_0(x) \overset{\text{Def. of interpolant } \Pi_h u_0}{\iff} \sum_{j=1}^{m} \underline{\xi_j(0)} \varphi_j(x) = \sum_{j=1}^{m} \underline{u_0(x_j)} \varphi_j(x)$$

$\Rightarrow \xi_j(0) = u_0(x_j)$ for $j = 1, 2, \dots, m$.

Looking at the above FE problem, we get

$$\sum_{j=1}^{m} \dot{\xi}_j(t) \underbrace{(\varphi_j, \varphi_i)_{L^2}}_{m_{ij}} + \sum_{j=1}^{m} \xi_j(t) \underbrace{(\varphi_j', \varphi_i')_{L^2}}_{s_{ij}} = \underbrace{(f(\cdot, t), \varphi_i)_{L^2}}_{F_i(t)} \text{ for } i = 1, \dots, m$$

which is the system of linear ODEs:

$$\text{(ODE)} \begin{cases} M \dot{\xi}(t) + S \xi(t) = F(t), \\ \xi(0) = \xi_0, \end{cases}$$

where $M = (m_{ij})_{i,j=1}^{m}$ mass matrix

$$M = \frac{h}{6} \begin{pmatrix} 4 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 4 \end{pmatrix}$$

$S = (s_{ij})_{i,j=1}^{m}$ stiffness matrix

$$S = \frac{1}{h} \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

$F(t) = (F_i(t))_{i=1}^{m}$ right-end side

$\xi_0 = \begin{pmatrix} u_0(x_1) \\ \vdots \\ u_0(x_m) \end{pmatrix}$ initial value

$\xi(t) = \begin{pmatrix} \xi_1(t) \\ \vdots \\ \xi_m(t) \end{pmatrix}$ unknown vector

(iv) Finally, we use a numerical method for IVP to find an approximation of $\xi(t)$ at discrete times.

Partition in time $[0,T]$: $t_0 = 0 < t_1 < t_2 < \dots < t_N = T$,

$t_n = n \cdot k$, $k = \frac{T}{N}$ is the <u>step-size</u>

Use implicit Euler to approximate $\zeta(t)$ at t_n:

$$M\left(\frac{\zeta^{(n+1)} - \zeta^{(n)}}{k}\right) + S\,\zeta^{(n+1)} = F(t_{n+1})$$

$$\Leftrightarrow \begin{cases} (M + kS)\,\zeta^{(n+1)} = M\zeta^{(n)} + kF(t_{n+1}) & \text{for } n = 0,1,2,\dots,N-1 \\ \zeta^{(0)} = \zeta_0 & \zeta^{(n)} \approx \zeta(t_n). \end{cases}$$

<u>Rem:</u>

- Solve a linear system ($Ax = b$) at <u>each</u> time step $n = 0, 1, 2, \dots$
- C-N would also be OK

  Explicit Euler (perhaps) less appropriate (stability)
- If integrals in $F(t_n)$ are not "easy"

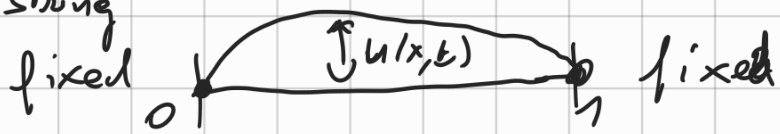
  $\leadsto$ use a quadrature formula.

Chapter IX: The wave equation

Goal: Study exact sol., present a discretisation of wave eq.

1) The wave equation in 1d:

Model problem: Consider a simple model for describing motion of a violon/guitare string



$$(W)\quad \begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = f(x,t) \\ u(0,t) = 0,\ u(1,t) = 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = v_0(x) \end{cases}$$

for $0 < x < 1$, $0 < t \le T$.

Th: For the case of a homogeneous wave equation ($f \equiv 0$), one has conservation of energy:

$$\underbrace{\underbrace{\frac{1}{2}\|u_t(\cdot,t)\|^2_{L^2(0,1)}}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}\|u_x(\cdot,t)\|^2_{L^2(0,1)}}_{\text{potential energy}}}_{\text{total energy}} = \overbrace{\frac{1}{2}\|u_0'\|^2_{L^2} + \frac{1}{2}\|v_0\|^2_{L^2}}^{\text{initial energy}} = \text{Constant} \quad \forall t \in [0,T].$$

Proof: • Multiply DE in (W) by u_t and by part:

$$\int_0^1 u_{tt}(x,t)\,u_t(x,t)\,dx - \underbrace{\int_0^1 u_{xx}(x,t)\,u_t(x,t)\,dx}_{\underbrace{u_x u_t\Big|_{x=0}^{1}}_{=0} + \int_0^1 u_x u_{tx}\,dx} = 0$$

$=0$ because of (BC)

$$\Leftrightarrow \int_0^1 \underbrace{u_{tt}(x,t)\,u_t(x,t)}_{\frac{1}{2}\frac{d}{dt}(u_t^2)}\,dx + \int_0^1 \underbrace{u_x(x,t)\,u_{tx}(x,t)}_{\frac{1}{2}\frac{d}{dt}(u_x^2)}\,dx = 0$$

$$\overset{\Leftrightarrow}{D} \quad \frac{1}{2}\frac{d}{dt}\left(\|u_t(\cdot,t)\|_{L^2}^2 + \|u_x(\cdot,t)\|_{L^2}^2 \right) = 0$$

Def norm

- Finally, we integrate $\int_0^t \dots ds$ and get

$$\frac{1}{2}\|u_t(\cdot,t)\|_{L^2}^2 + \frac{1}{2}\|u_x(\cdot,t)\|_{L^2}^2 = \frac{1}{2}\|u_0'\|_{L^2}^2 + \frac{1}{2}\|v_0\|_{L^2}^2$$

:-) ■

By introducing a new variable for the velocity,

$v := u_t$, we can rewrite (W) as a system of
$v(x,t) = u_t(x,t)$
DE:

$$\begin{cases} u_t = v \\ v_t = u_{tt} = u_{xx} + f \end{cases} \iff \underbrace{\begin{pmatrix} u \\ v \end{pmatrix}_t}_{W_t} = \underbrace{\begin{pmatrix} 0 & 1 \\ \partial_x^2 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u \\ v \end{pmatrix}}_{W} + \underbrace{\begin{pmatrix} 0 \\ f \end{pmatrix}}_{F}$$

$\iff W_t = AW + F$ (syst. of DE)

This can be used to derive a $c(0,1) - c(0,1)$ discretisation $\rightsquigarrow$ book p. 194 (if interested)

2) Discretisation of the wave equation:

We shall apply FEM in space and C-N in time to find a numerical approximation of sol. (W).

(i) Find variational formulation:

Consider $H_0^1(0,1)$ and test against $v \in H_0^1(0,1)$.

Obtain:

$$(VF)\begin{cases} \text{For } 0 < t \leq T, \text{ find } u(\cdot, t) \in H_0^1 \text{ s.t.} \\ (u_{tt}(\cdot,t), v)_{L^2} + (u_x(\cdot,t), v')_{L^2} = (f(\cdot,t), v)_{L^2} \quad \forall v \in H_0^1 \\ u(x,0) = u_0(x), \; u_t(x,0) = v_0(x) \end{cases}$$