Assignment 2

February 15, 2021

Instructions: You may work in a group of 2 persons but hand in only one report for the group (with both names and relevant information, .m files separately). Each of the 5 tasks below give one point. To pass the assignment, one needs at least two points. Submit a concise report (max 10 pages) by March 08 to Malin Malin (submitted report with less than two points may be resubmitted one time, without the possibility of getting bonus points). The possibly obtained extra point(s) (at most three points) will be counted as bonus point(s) for the written examination.

The assignment is based on a previous version from 2020 *by M. Asadzadeh.*

Tasks

1. Solve all exercises (a)-(c) of the theoretical part.

Choose one of the applications from the list below and do the following:

2. State a model problem that describes this application and motivate your model physically. In addition, derive the corresponding weak formulation/variational formulation.
3. Describe the discretisation procedure from the triangulation of the domain to the final linear system of equations. Don't forget to explicitly specify the used matrices.
4. Solve the model problem numerically, explain and plot your results. You may use MATLAB's PDE TOOLBOX/PDE MODELER or similar if needed.
5. Derive a stability analysis as well as error estimates. You may use a simplified model if needed (by looking at homogeneous Dirichlet boundary conditions for instance).

Theoretical part

Consider a nice domain Ω (in $\mathbb{R}^2$ for (a)-(b) and in $\mathbb{R}^1$ for (c)) and a given initial value u_0. Consider the linear heat equation

$$\begin{aligned} u_t(x) - \Delta u(x) &= 0, \qquad x \in \Omega, \quad t > 0, \\ u(x,t) &= 0, \qquad x \in \partial\Omega, \quad t > 0, \\ u(x,0) &= u_0(x), \qquad x \in \Omega. \end{aligned}$$

(a) Show the following stability estimates:

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u(\cdot,s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \le \|u_0\|^2_{L^2(\Omega)} \qquad \text{for} \quad t>0,$$

$$\|\Delta u(\cdot,t)\|_{L^2(\Omega)} \le \frac{1}{t}\|u_0\|_{L^2(\Omega)} \qquad \text{for} \quad t>0.$$

The latter estimate is the so-called parabolic smoothing estimate (or strong stability). This estimate describes the fact that the solution to the above linear heat equation immediately becomes smoother than the initial data u_0.

(b) How do these estimates change if one replaces $\Delta u(x)$ with $u_{x_1x_1} + 4u_{x_2x_2}$?

(c) Solve the above heat equation on $\Omega = [0,1]$ using separation of variables and a Fourier series. Study how fast the Fourier coefficients of the exact solution decay. Finally, show the above smoothing property by considering the Fourier series representation of your obtained solution.

List of applications

The goal of this part is to solve an application of interest using a FEM implementation. Start by selecting one of the applications ((i)-(v)) below. Implement a FEM method in a software of your choice, evaluate and plot your results. Finally, draw some conclusions concerning the exact solution to your model as well as the numerical one.

You thus have to think of:

- An interesting real world problem.
- A mathematical modelling including for instance the choice of boundary conditions or truncation of the computational domain in case of unbounded domains.
- Computational aspects.
- Analytical aspects, seek to simplify the model so that it is possible to obtain an analytical solution. Solve the simplified model and think about the extra assumptions you have made, are they realistic?

Choices for applications (the notation $x = (x_1, x_2) \in \mathbb{R}^2$ is used below):

(i) **Convection-diffusion-absorption/reaction.** Let $\alpha, \beta, \varepsilon, f$ be nice functions. Consider a $2d$ convection-diffusion-absorption/reaction problem of the form

$$\alpha u + \beta \cdot \nabla u - \nabla \cdot (\varepsilon \nabla u) = f,$$

together with suitable boundary conditions on the boundary Γ of $\Omega \subset \mathbb{R}^2$. Here $u = u(x)$ is an unknown concentration, $\varepsilon = \varepsilon(x)$ is a given (small) diffusion coefficient,

$\beta = \beta(x)$ is a given velocity field, $\alpha = \alpha(x)$ is a given absorption/reaction coefficient and $f = f(x)$ is a given production term. Solve a convection-dominated problem of this form for instance related to pollution control, where f is a Dirac delta function at some point $P \in \Omega$. Determine for instance the width of the "smoke plume".

(ii) **Electrostatics.** Let ε, ρ be nice functions. Consider the basic problem of $2d$ electrostatics

$$\begin{aligned} \nabla \cdot (\varepsilon E) &= \rho, \\ E &= -\nabla \phi, \end{aligned}$$

together with suitable boundary conditions corresponding to a part of the boundary of $\Omega \subset \mathbb{R}^2$ being a perfect conductor and the remaining part being insulated. Here $E = E(x)$ is the electric field, $\phi = \phi(x)$ the electric potential, $\varepsilon = \varepsilon(x)$ the dielectricity coefficient, and $\rho = \rho(x)$ the charge density. Solve a problem of this form first on a simple domain and then if possible on a domain with a boundary containing a sharp non-convex corner. Study the behavior of the electric field in the vicinity of the corner.

(iii) **2d fluid flow.** The velocity $u = (u_1, u_2)$ of an incompressible irrotational $2d$ fluid may be expressed through a potential ϕ by $u = \nabla\phi$. Coupled with the incompressibility equation $\nabla \cdot u = 0$ this gives the Laplace equation for ϕ:

$$\nabla \cdot (\nabla \phi) = \Delta \phi = 0,$$

together with suitable boundary conditions expressing for instance that $u \cdot n = 0$ on solid boundaries. Note that it is not possible to use Neumann boundary conditions on the entire boundary. Solve for example a problem of the following type:

1. flow through a $2d$ nozzle
2. flow around a disc or wing profile.

Use the gradient plot function to visualise the flow.

(iv) **Heat conduction.** Let κ and f be given nice functions. Consider a $2d$ stationary heat equation

$$\nabla \cdot q = f, \quad q = -\kappa \nabla u,$$

on a domain $\Omega \subset \mathbb{R}^2$ with suitable boundary conditions. Here, $u = u(x)$ is the temperature, $q = q(x)$ the heat flow, $\kappa = \kappa(x)$ the heat conduction coefficient and $f = f(x)$ a given production term. Solve for instance a problem of this form modeling a hot water pipe buried in a half space and determine the temperature on the boundary of the half space above the pipe using a Robin boundary condition on the surface.

(v) **Quantum physics.** Consider the $2d$ stationary Schrödinger eigenvalue problem

$$-\frac{\hbar^2}{2m} \Delta u + V(x) u = \lambda u,$$

where V is a given potential (choose a nonconstant function), $\hbar$ is Planck's constant divided by 2π and m is the particle mass. The goal is then to find eigenpairs $\{\lambda, u\}$ consisting of an eigenvalue and an eigenfunction. Give a quantum physical interpretation of the eigenvalues and corresponding eigenfunctions determined by the above problem. Normalize the constants and solve the problem for some suitable domain and potential (rectangular domain or circle, harmonic potential $V(x) = x_1^2 + x_2^2$, Coulomb potential, double-well potential for instance). Discuss your computational results from a quantum physical viewpoint.