2021-02-17 Exercise session w5 Malin Nilsson

21) Consider an IVP:

$$\begin{cases} \dot{u}(t) + a(t)u(t) = f(t), & 0 < t \leq T \\ u(0) = u_0 \end{cases}$$

Assume $\int_{I_j} f(s)\,ds = 0$, $j = 1, 2, \dots$

$I_j = (t_{j-1}, t_j)$, $t_j = jk$, $k > 0$

Prove that if $a(t) \geq 0$, then u satisfies:

$$|u(t)| \leq e^{-A(t)}|u_0| + \max_{0 \leq s \leq t} |kf(s)|$$

Solution. From thm 6.1, we have:

$$u(t) = u_0 e^{-A(t)} + \int_0^t e^{-(A(t)-A(s))} f(s)\,ds$$

$a(t) = A'(t) \geq 0 \Rightarrow A(t) \geq A(s)$

$\Rightarrow e^{-(A(t)-A(s))} \leq 1$

$\Rightarrow |u(t)| \leq |u_0| e^{-A(t)} + \left| \int_0^t f(s)\,ds \right|$

Let $t \in I_j$. Then we have

$$\left| \int_0^t f(s)\,ds \right| = \Bigg| \underbrace{\sum_{i=1}^{j-1} \int_{I_i} f(s)\,ds}_{=0} + \int_{t_{j-1}}^t f(s)\,ds \Bigg| =$$

$$= \left| \int_{t_{j-1}}^t f(s)\,ds \right| \leq k \|f\|_{L^\infty(0,t)}$$

$$\Rightarrow |u(t)| \le e^{-A(t)}|u_0| + \max_{0 \le s \le t} |kf(s)| . \quad //.$$

22) Compute the cG(1) approx. for the IVP

$$\begin{cases} \dot{u}(t) + a(t)u(t) = f(t), & 0 < t \le T \\ u(0) = u_0 \end{cases}$$

for a) $a(t) = 4$, $f(t) = t^2$, $u_0 = 1$

b) $a(t) = -t$, $f(t) = t^2$, $u_0 = 1$

Solution. Divide $(0,T)$ into equidistant sub-intervals (t_{n-1}, t_n), $t_n - t_{n-1} = k$

cG(1) method for IVP:s : trial space = p.w. linear functions, basis: usual hat fcns.

Test space: p.w. constant fcns. Basis: $v \equiv 1$ on each interval.

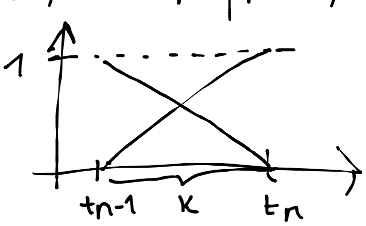
Multiply DE by v and integrate:

a) $$\int_{t_{n-1}}^{t_n} \dot{u}(t)\,dt + 4\int_{t_{n-1}}^{t_n} u(t)\,dt = \int_{t_{n-1}}^{t_n} t^2\,dt \qquad (*) \quad n = 1, \dots, N$$

On an interval $(t_{n-1}, t_n]$, $u(t)$ is approxim. by a p.w. linear fcn:

$$\int_{t_{n-1}}^{t_n} u(t)\,dt \approx \int_{t_{n-1}}^{t_n} \left(u_{n-1}\varphi_{n-1}(t) + u_n \varphi_n(t) \right) dt =$$

φ_i usual hat fcn:



$$= \dots = \frac{k}{2}(u_{n-1}+u_n)$$

$$\int_{t_{n-1}}^{t_n} t^2\,dt = \dots = \frac{1}{3}(t_n^3 - t_{n-1}^3) = \left\{ \begin{array}{l} t_n = nk \\ (nk)^3 - (n-1)^3k^3 = \\ k^3(n-n+1) = k^3 \end{array} \right\} = \frac{k^3}{3}$$

This is of course WRONG! should be $\boxed{= \frac{3n^2-3n+1}{3}k^3}$

$$(*) \Rightarrow u_n - u_{n-1} + 2k(u_{n-1}+u_n) = \cancel{\frac{k^3}{3}}$$

$$(1+2k)u_n = (1-2k)u_{n-1} + \cancel{\frac{k^3}{3}}$$

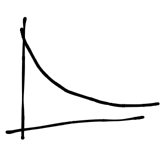
$$u_n = \frac{1-2k}{1+2k}\left(u_{n-1} + \cancel{\frac{k^3}{3(1-2k)}}\right) =$$

Replace all $\frac{k^3}{3}$

$$= \frac{1-2k}{1+2k}\left(\frac{1-2k}{1+2k}\left(u_{n-2} + \cancel{\frac{k^3}{3(1-2k)}}\right) + \cancel{\frac{k^3}{3(1-2k)}}\right) =$$

$$= \dots = \left(\frac{1-2k}{1+2k}\right)^n u_0 + \sum_{j=0}^{n-1}\left(\frac{1-2k}{1+2k}\right)^{n-j}\cancel{\frac{k^3}{3(1-2k)}} =$$

$$= \{u_0 = 1\} = \underbrace{\left(\frac{1-2k}{1+2k}\right)}_{<1}^{n} + \sum_{j=0}^{n-1}\underbrace{\left(\frac{1-2k}{1+2k}\right)}_{<1}^{n-j}\cancel{\frac{k^3}{3(1-2k)}}$$



cf. thm 6.2a) (prob. 21) $a > 0$

b) $a(t) = -t$. Multiply DE by test fcn., integrate:

$$\int_{t_{n-1}}^{t_n} u(t)\,dt - \underbrace{\int_{t_{n-1}}^{t_n} t\,u(t)\,dt}_{(**)} = \int_{t_{n-1}}^{t_n} t^2\,dt \qquad (*) \quad n = 1, \dots, N$$

$$(**) = \int_{t_{n-1}}^{t_n} t\left(u_{n-1}\varphi_{n-1}(t) + u_n\varphi_n(t)\right)dt = \begin{cases} \varphi_{n-1} = \frac{t_n - t}{k}, & (t_{n-1}, t_n) \\ \varphi_n = \frac{t - t_{n-1}}{k}, & -\cdot- \end{cases}$$

$$= \frac{1}{k}\int_{t_{n-1}}^{t_n}\left(u_{n-1}(tt_n - t^2) + u_n(t^2 - tt_{n-1})\right)dt = \dots =$$

$$= -\frac{k}{6}\left(2u_{n-1}k - 3u_{n-1}t_n + u_nk - 3u_nt_n\right) =$$

$$= \{t_n = nk\} = u_{n-1}\left(\frac{1}{2}nk^2 - \frac{1}{3}k^2\right) + u_n\left(\frac{1}{2}nk^2 - \frac{1}{6}k^2\right)$$

$$(*) \Rightarrow u_n - u_{n-1} - u_{n-1}\left(\frac{1}{2}nk^2 - \frac{1}{3}k^2\right) - u_n\left(\frac{1}{2}nk^2 - \frac{1}{6}k^2\right) = \cancel{\frac{k^3}{3}}$$

$$u_n\left(1 + \frac{1}{6}k^2 - \frac{1}{2}nk^2\right) - u_{n-1}\left(1 + \frac{1}{2}nk^2 - \frac{1}{3}k^2\right) = \cancel{\frac{k^3}{3}}$$

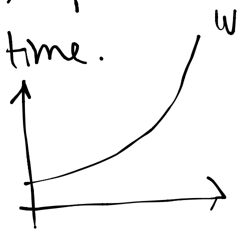
$$\Rightarrow u_n = \frac{1 + k^2\left(\frac{n}{2} - \frac{1}{3}\right)}{1 - k^2\left(\frac{n}{2} - \frac{1}{6}\right)}\left(u_{n-1} + \cancel{\frac{k^3}{3\left(1 + k^2\left(\frac{n}{2} - \frac{1}{3}\right)\right)}}\right) = \dots =$$

$$= \underbrace{\left(\frac{1 + k^2\left(\frac{n}{2} - \frac{1}{3}\right)}{1 - k^2\left(\frac{n}{2} - \frac{1}{6}\right)}\right)^n}_{>1} + \sum_{j=0}^{n-1}\underbrace{\left(\frac{1 + k^2\left(\frac{n}{2} - \frac{1}{3}\right)}{1 - k^2\left(\frac{n}{2} - \frac{1}{6}\right)}\right)^{n-j}}_{>1}\cancel{\frac{k^3}{3\left(1 + k^2\left(\frac{n}{2} - \frac{1}{3}\right)\right)}}$$

Decreases w/ time

perturbations will increase w/ time.

$a(t) = -t.$



23) Consider the discontinuous Galerkin method $dG(0)$ for the IVP

$$\begin{cases} \dot{u}(t) + au(t) = 0 & 0 < t \leq T, \quad a \geq 0 \\ u(0) = u_0 \end{cases}$$

Prove the stability estimate

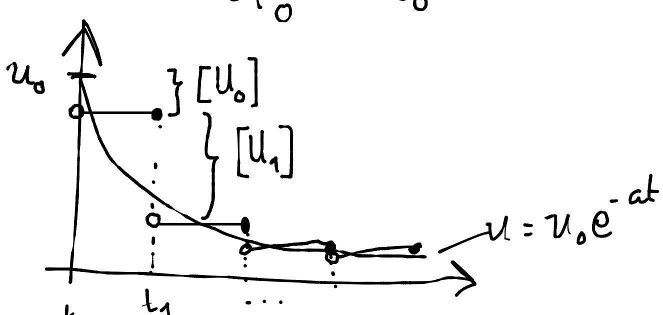
$$|U_N|^2 + \sum_{n=0}^{N-1} |[U_n]|^2 \leq |u_0|^2$$

<u>Solution.</u>

$$[U_n] = U_n^+ - U_n^- = U_{n+1} - U_n$$

$$U_n^+ = \lim_{t \searrow t_n} U(t) = U\big|_{(t_n, t_{n+1}]} = U_{n+1}$$

$$U_0 = u_0$$



$$0 = t_0 < t_1 < \ldots < t_N = T$$

$$t_j - t_{j-1} = k$$

$dG(0)$ Formulation: Find $U \in W_k^{(0)} = \{v : v \text{ p.w. const}\}$:

$$(*) \quad \sum_{n=1}^{N} \left(\int_{t_{n-1}}^{t_n} (\dot{U} + aU) v \, dt + [U_{n-1}] v_{n-1}^+ \right) = 0$$

for all $v \in W_k^{(0)}$

For U p.w. const. $\Rightarrow \dot{U}\big|_{(t_{n-1}, t_n)} = 0$

$U = \sum U_n 1_{I_n}$ ← (indicator fcn)

Basis: v_n s.t $v_n = \begin{cases} 1 & \text{on } I_n \\ 0 & \text{otherwise} \end{cases} = 1_{I_n}$

$$(*) \Rightarrow \int_{t_{n-1}}^{t_n} a U_n \, dt + (U_n - U_{n-1}) = 0 \quad n = 1, \dots, N$$

$$\Rightarrow akU_n + U_n - U_{n-1} = 0 \qquad n = 1, \dots, N$$

Multiply by U_n

$$akU_n^2 + U_n^2 - U_n U_{n-1} = 0 \Rightarrow$$

$$U_n^2 - U_n U_{n-1} = -akU_n^2 \leq 0$$

$$\Rightarrow 0 \geq \sum_{n=1}^{N} (U_n^2 - U_n U_{n-1}) =$$

$$= \sum_{n=1}^{N} (U_n^2 - U_n U_{n-1}) + \frac{1}{2} U_0^2 - \frac{1}{2} u_0^2 =$$

$$= \frac{1}{2} \sum_{n=1}^{N} U_n^2 + \frac{1}{2} \sum_{n=1}^{N} (U_n^2 - 2U_n U_{n-1}) + \frac{1}{2} U_0^2 - \frac{1}{2} u_0^2 =$$

$$= \frac{1}{2} U_N^2 + \frac{1}{2} \sum_{n=1}^{N} (U_n^2 - 2U_n U_{n-1} + U_{n-1}^2) - \frac{1}{2} u_0^2 =$$

$$= \frac{1}{2} U_N^2 + \frac{1}{2} \sum_{n=1}^{N} (U_n - U_{n-1})^2 - \frac{1}{2} u_0^2 =$$

$$= \frac{1}{2} U_N^2 + \frac{1}{2} \sum_{n=1}^{N} [U_{n-1}]^2 - \frac{1}{2} u_0^2$$

$$\Rightarrow \frac{1}{2}|U_N|^2 + \frac{1}{2}\sum_{n=1}^{N}[U_{n-1}]^2 \le \frac{1}{2}|u_0|^2$$

$$|U_N|^2 + \sum_{n=0}^{N-1}[U_n]^2 \le |u_0|^2 \quad \text{//}.$$

20) Consider the BVP

$$\begin{cases} -\varepsilon u'' + x u' + u = f & \text{in } I = (0,1) \\ u(0) = u'(1) = 0 \end{cases}$$

$\varepsilon > 0$ const, $f \in L^2(I)$. Prove that

$$\|\varepsilon u''\|_{L^2(I)} \le \|f\|_{L^2(I)}$$

Solution.

$$\|\varepsilon u''\|_{L^2}^2 = \|x u' + u - f\|_{L^2}^2 =$$

$$= \int_0^1 (\underbrace{xu' + u} - \underbrace{f})^2 \, dx =$$

$$= \underbrace{\int_0^1 (xu' + u)^2 dx - 2\int_0^1 (xu' + u) f dx}_{(**)} + \underbrace{\int_0^1 f^2 dx}_{= \|f\|_{L^2}^2}$$

Show that $(**) \le 0$

$$f = -\varepsilon u'' + x u' + u \Rightarrow$$

$$(**) = \int_0^1 (xu'+u)^2 + 2\varepsilon \int_0^1 (xu'+u)u'' - 2\int_0^1 (xu'+u)^2 =$$

$$= 2\varepsilon \int_0^1 (xu'+u)u''\,dx - \int_0^1 (xu'+u)^2\,dx =$$

$$= \left\{ \begin{aligned} &\text{P.I. } \int_0^1 (xu'+u)u'' = \left[(xu'+u)u'\right]_0^1 - \int_0^1 (u'+xu''+u')u' \\ &= -\int_0^1 (2u'+xu'')u' = -\int 2(u')^2 - \int xu'u'' \end{aligned} \right\}$$

$$= -4\varepsilon \int_0^1 (u')^2 - 2\varepsilon \int_0^1 xu'u'' - \int (xu'+u)^2 =$$

$$= \left\{ \begin{aligned} &\int xu'u'' = \left[xu'u'\right]_0^1 - \int_0^1 (u'+xu'')u' = \\ &= -\int (u')^2 - \int xu'u'' \Rightarrow \\ &2\int xu'u'' = -\int (u')^2 \end{aligned} \right\}$$

$$= -4\varepsilon \|u'\|_{L^2}^2 + \varepsilon \|u'\|_{L^2}^2 - \|xu'+u\|_{L^2}^2 =$$

$$= -3\varepsilon \|u'\|_{L^2}^2 - \|xu'+u\|_{L^2}^2 \leq 0$$

$$\Rightarrow \quad \|\varepsilon u''\|_{L^2}^2 \leq \|f\|_{L^2}^2 .$$