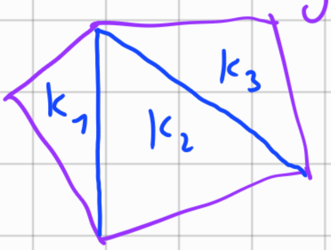
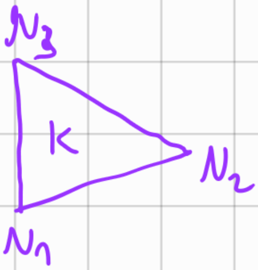
Recall:

- 

$\Omega \subset \mathbb{R}^2$ with polygonal $\partial\Omega$

mesh/triangulation $T_h = \{K_1, K_2, K_3\}$

- 

$P_1(K) = \{v : K \to \mathbb{R} ;\ v(x_1, x_2) = c_0 + c_1 x_1 + c_2 x_2\}$

- $V_h = \{ v \in \mathcal{C}^0(\Omega) ;\ v_{|K} \in P_1(K) \quad \forall K \in T_h \}$

$= \mathrm{span}\left(\{\varphi_j\}_{j=1}^{n_p}\right)$, n_p = nbr of nodes in T_h

$\underset{V_h}{\underset{\cap}{v}} = \sum_{j=1}^{n_p} \alpha_j \varphi_j$

- Linear interpolant of f on triangle K:

$$\pi_1 f = \sum_{j=1}^{3} f(N_j)\, \varphi_j \in P_1(K)$$

$$\| f - \pi_1 f \|_{L^2(K)} \leq C_K \cdot h_K^2 \cdot \| f \|_{H^2(K)}$$

- Continuous pw linear interpolant of f

$$\pi_h f = \sum_{j=1}^{n_p} f(N_j)\, \varphi_j \in V_h$$

Th: For $f \in H^2(\Omega)$ and $\Pi_h f$ as above, then:

$$\| f - \Pi_h f \|^2_{L^2(\Omega)} \leq C \sum_{K \in T_h} h_K^4 \, \| f \|^2_{H^2(K)}$$

$$\| \nabla (f - \Pi_h f) \|^2_{L^2(\Omega)} \leq C \sum_{K \in T_h} h_K^4 \, \| f \|^2_{H^2(K)} .$$

Proof:

$$\| f - \Pi_h f \|^2_{L^2(\Omega)} = \sum_{K \in T_h} \| f - \underset{\substack{\cap \\ P_1(K)}}{\Pi_h f} \|^2_{L^2(K)} \leq$$

$$\underset{\substack{\uparrow \\ \text{Previous theorem}}}{\leq} \sum_{K \in T_h} C_K^2 \, h_K^4 \, \| f \|^2_{H^2(K)} \leq$$

$$\underset{\substack{\uparrow \\ \text{For "nice" triangulation, one has } C_K \leq C \ \forall K \in T_h}}{\leq} C^2 \sum_{K \in T_h} h_K^4 \, \| f \|^2_{H^2(K)} \leq$$

$$\underset{\substack{\uparrow \\ h_K \leq h \ \forall K \in T_h}}{\leq} C^2 \cdot h^4 \sum_{K \in T_h} \| f \|^2_{H^2(K)} \leq C^2 \cdot h^4 \, \| f \|^2_{H^2(\Omega)}$$

$\blacksquare$

What about higher order approximation?
(linear $\leftrightarrow$ polyn. of degree $2, 3, 4, \dots$)

Rem: Consider a cont. pw polynomial interpolant of degree $r-1$, then one has

$$\|\pi_h f - f\|_{L^2(\Omega)} \leq C \cdot h^r \cdot \|f\|_{H^r(\Omega)}$$

for $f \in H^r(\Omega)$

8) L^2-projection:

Def: Given a function $f \in L^2(\Omega)$, the L^2-projection $P_h f \in V_h$ of f is defined by

$$\int_\Omega (f - P_h f)\, v \, dx = 0 \quad \forall\, v \in V_h$$

$$\Updownarrow$$

$$(f - P_h f, v)_{L^2(\Omega)} = 0 \quad \forall\, v \in V_h \quad \Leftrightarrow$$

error $f - P_h f \perp V_h$

One has that $P_h f$ is the best approximation of f in V_h with respect to the L^2-norm:

$$\|P_h f - f\|_{L^2(\Omega)} \leq \|v - f\|_{L^2(\Omega)} \quad \forall\, v \in V_h$$

In addition, one has

$$\|P_h f - f\|_{L^2(\Omega)} \le \|\Pi_h f - f\|_{L^2(\Omega)} \le C\cdot h^r \cdot \|f\|_{H^r(\Omega)} \qquad \forall f \in H^r,\ r=2,3.$$

V_h

previous remark / theorem

Proof:

$$\|P_h f - f\|^2_{L^2(\Omega)} = (P_h f - f,\ P_h f - v + v - f)_{L^2(\Omega)} =$$

Def L^2-norm, $v \in V_h$

$$= (P_h f - f,\ \overbrace{P_h f - v}^{\in V_h})_{L^2(\Omega)} + (P_h f - f,\ v - f)_{L^2(\Omega)} =$$

linearity of $(\cdot,\cdot)$

$$= 0 + (P_h f - f,\ v - f)_{L^2(\Omega)} =$$

Def of L^2-projection

$$= (P_h f - f,\ v - f)_{L^2(\Omega)} \overset{C-S}{\le} \|P_h f - f\|_{L^2(\Omega)} \cdot \|v - f\|_{L^2(\Omega)}$$

$$(*) \quad \|P_h f - f\|_{L^2(\Omega)} \le \|v - f\|_{L^2(\Omega)} \quad \forall v \in V_h \qquad \blacksquare$$

$$\|P_h f - f\|_{L^2}^{\cancel{2}} \le \cancel{\|P_h f - f\|_{L^2}} \cdot \|v - f\|_{L^2}$$

Chapter XI: FEM for Poisson's eq in 2d

Goal: Derive FEM for Poisson, give error estimates.

($1d$: $-u'' = f$)

Recall: Poisson's eq.
$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^2 \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(VF) Find $u \in H_0^1(\Omega)$ s.t.
$$(\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega)$$

1) FE approximation:

Let T_h be a triangulation of Ω and V_h be the space of cont. pw linear functions on T_h.

We consider the space V_h^0, defined as

$V_h^0 = \{ v \in V_h : v_{|\partial\Omega} = 0 \}$

The FE problem thus reads:

(FE) Find $u_h \in V_h^0$ s.t. $(\nabla u_h, \nabla \chi)_{L^2(\Omega)} = (f, \chi)_{L^2(\Omega)}$
$\forall \chi \in V_h^0$.

To get a linear system of eq. from (FE), we observe that

$V_h^0 = \mathrm{span}(\{\varphi_j\}_{j=1}^{n_i})$, where n_i is the number of interior nodes.

Hence, one can write

$u_h = \sum_{j=1}^{n_i} \underset{\text{unknown coeff.}}{\xi_j} \varphi_j$ and we take $\chi = \varphi_i$

for $i = 1, \dots, n_i$.

Inserting the above into (FE) gives us:

$$\sum_{j=1}^{n_i} \xi_j \underbrace{(\nabla \varphi_j, \nabla \varphi_i)_{L^2(\Omega)}}_{s_{ij}} = \underbrace{(f, \varphi_i)_{L^2(\Omega)}}_{b_i} \quad \forall i = 1, \dots, n_i$$

which is equivalent to the linear system

$$S' \cdot \xi = b \quad , \text{ where}$$

$S' = (s_{ij})_{i,j=1}^{n_i}$ ⇝ stiffness matrix

$b = (b_i)_{i=1}^{n_i}$ ⇝ load vector

$\xi = (\xi_j)_{j=1}^{n_i}$ ⇝ unknow coeff.

2) Notes on implementation:

Consider $\Omega \subset \mathbb{R}^2$ with polygonal $\partial\Omega$ and a triangulation $T_h = \{K\}$, triangle K.

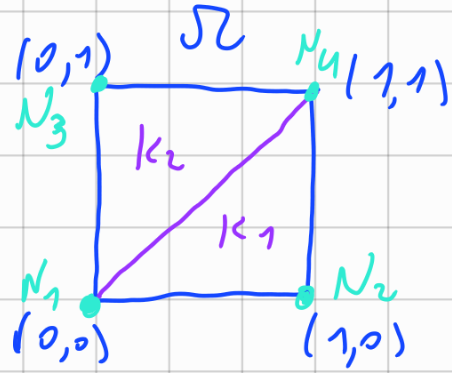
a) Data structure for a triangulation:

The standard way of representing a triangle mesh with n_p nodes and n_E elements on a computer is using 2 matrices:

P ⇝ point matrix

$T \rightsquigarrow$ connectivity matrix

Ex:



$\Omega \to$ unit square

One has

$$P = \begin{array}{c} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array} \begin{array}{l} \text{node } N_1 \\ \text{node } N_2 \\ \text{node } N_3 \\ \text{node } N_4 \end{array}$$

$\rightsquigarrow$ list all nodes in T_h

($P \rightsquigarrow n_p \times 2$)

One has

$$T = \begin{pmatrix} 1 & 2 & 4 & \vdots & 1 & 1 & 0 \\ 1 & 4 & 3 & \vdots & 0 & 1 & 1 \end{pmatrix}$$

2 triangles with nodes

$1,2,4 \rightsquigarrow K_1$

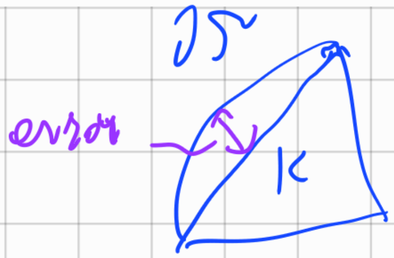
$1,4,3 \rightsquigarrow K_2$

indicates which side of a triangle is a real boundary of Ω

$1 \to$ real boundary

$0 \to$ not real boundary

($T \rightsquigarrow n_t \times (3+3)$)



b) Assembly of the matrices:

Once a triangle mesh is generated, one can compute the stiffness matrix and the mass matrix.

Idea: For the stiffness matrix S', one needs to compute the entries

$$\int_\Omega \nabla\varphi_i\cdot\nabla\varphi_j\,dx = \sum_{\ell=1}^{nt} \underbrace{\int_{K_\ell}\nabla\varphi_i\cdot\nabla\varphi_j\,dx}_{=:(\nabla\varphi_i,\nabla\varphi_j)_{K_\ell}}$$

In a FE code, one first computes the element stiffness matrix S_i' on each element K_i with vertices $\vec{x}_j, \vec{x}_k, \vec{x}_\ell$ by computing:

$$S_i = \begin{pmatrix} (\nabla\varphi_j, \nabla\varphi_j)_{K_i} & (\nabla\varphi_j, \nabla\varphi_k)_{K_i} & (\nabla\varphi_j, \nabla\varphi_l)_{K_i} \\ & (\nabla\varphi_k, \nabla\varphi_k)_{K_i} & (\nabla\varphi_k, \nabla\varphi_l)_{K_i} \\ \text{Symmetric} & & (\nabla\varphi_l, \nabla\varphi_l)_{K_i} \end{pmatrix} =$$

$$= \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ & p_{22} & p_{23} \\ \text{Sym.} & & p_{33} \end{pmatrix}$$

One then adds this element contribution S_i at the appropriate location j, k, l in the global stiffness matrix S:

$$S = S + \begin{array}{cc} \begin{pmatrix} 0 & & & & & & 0 \\ & p_{11} & & p_{12} & & p_{13} & \\ & & 0 & & 0 & & \\ & p_{12} & & p_{22} & & p_{23} & \\ & & 0 & & 0 & & \\ 0 & p_{13} & & p_{23} & & p_{33} & 0 \end{pmatrix} & \begin{array}{c} \\ j \\ \\ k \\ \\ l \end{array} \\ \begin{array}{ccccccc} & j & & k & & l & \end{array} & \end{array}$$

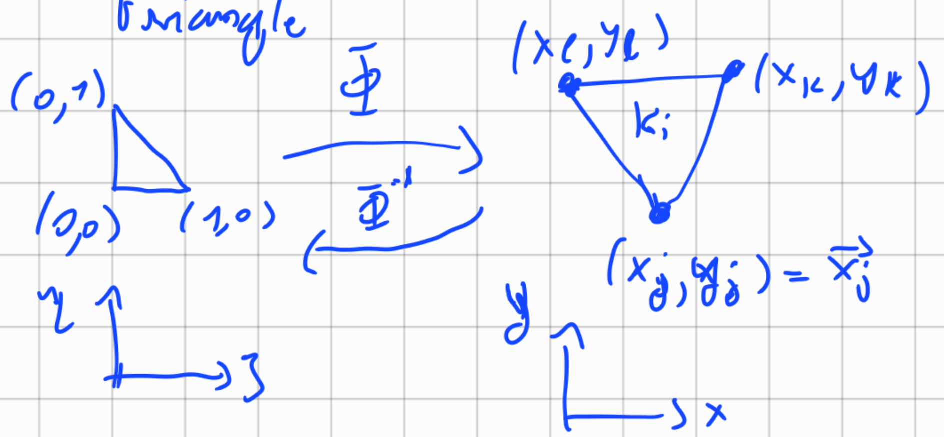
This procedure is called the assembly process.

c) Computation of the element stiffness matrix / mass matrix:

To compute the element stiffness matrix S_i, one introduces a linear map:

$$\Phi_i : \hat{K} \longrightarrow K_i$$

reference triangle $\longrightarrow$ element triangle K_i



$$(0,0) \longmapsto (x_j, y_j)$$

$$(0,1) \longmapsto (x_\ell, y_\ell)$$

$$(1,0) \longmapsto (x_k, y_k)$$