2021-02-24 Exercise Session w.6 Malin Nilsson

34) $-\Delta E = \delta_0$ in $\mathbb{R}^d$ $(*)$

$\delta_0$ = dirac delta distribution

Then $E$ is called the <u>fundamental solution</u> to $-\Delta$

$(*) \Leftrightarrow$ for any $v \in C_0^\infty$, $E$ satisfies

$$-\int_{\mathbb{R}^d} \Delta E(x) v(x) = v(0)$$

Prove that $E(x) = \frac{1}{2\pi} \log\left(\frac{1}{|x|}\right)$

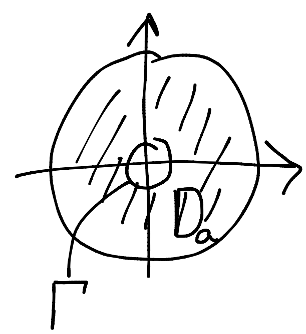
is a fundamental solution to $-\Delta$ in $\mathbb{R}^2$.

<u>Solution.</u> Want to show:

$$-\int_{\mathbb{R}^2} \Delta\left(\frac{1}{2\pi} \log\left(\frac{1}{|x|}\right)\right) v(x) = v(0) \quad \forall v \in C_0^\infty$$

Fix $v \in C_0^\infty$

Let $D_a = \left\{x \in \mathbb{R}^2 : 0 < a < |x| < \frac{1}{a}\right\}$




$\Gamma := \{x \in \mathbb{R}^2 : |x| = a\}$

$a$ s.t. $\operatorname{supp} v \subset \left\{x \in \mathbb{R}^2 : |x| < \frac{1}{a}\right\}$

i.e. $v$ vanishes outside of $|x| = \frac{1}{a}$

$$-\int_{D_a} \Delta E v = \left\{ \begin{array}{l} \text{Green's:} \\ \int_\Omega \Delta u v = \int_{\partial\Omega} (\nabla u \cdot n) v - \int_\Omega \nabla u \cdot \nabla v \end{array} \right\} =$$

$$= -\int_{\partial D_a} (\nabla E \cdot n) v + \int_{D_a} \nabla E \cdot \nabla v = \{\text{Green's}\} =$$

$$= -\int_{\partial D_a} (\nabla E \cdot n) v + \int_{\partial D_a} (\nabla v \cdot n) E - \int_{D_a} E \Delta v =$$

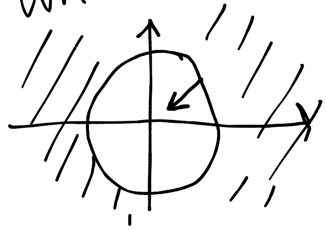
$= \{v$ vanishes before $|x| = \frac{1}{a}\} =$

$$= -\int_{\Gamma} (\nabla E \cdot n) v + \int_{\Gamma} (\nabla v \cdot n) E - \int_{D_a} E \Delta v$$

$\{$Exercise 27: $-\Delta E = 0, \quad x \neq 0 \Rightarrow -\int_{D_a} \Delta E v = 0\}$

$$\Rightarrow \int_{D_a} E \Delta v = \int_{\Gamma} (\nabla v \cdot n) E - \int_{\Gamma} (\nabla E \cdot n) v$$

What is $\nabla E \cdot n$ on $\Gamma$?



$|x| = a \qquad \nabla E = -\frac{1}{2\pi} \frac{x}{|x|^2}$

$|n| = 1$

$\Rightarrow n = \frac{-x}{a}$

$$\Rightarrow \nabla E \cdot n = -\frac{1}{2\pi} \frac{x}{|x|^2} \cdot \frac{-x}{a} = \frac{1}{2\pi a} \text{ on } \Gamma.$$

$$\Rightarrow \int_{D_a} E \Delta v = \int_{\Gamma} \frac{1}{2\pi} \log\left(\frac{1}{|x|}\right) \nabla v \cdot n -$$

$$- \int_{\Gamma} \frac{1}{2\pi a} v =$$

$$= \frac{1}{2\pi}\log\left(\frac{1}{a}\right)\underbrace{\int_{\Gamma}\nabla v\cdot n}_{\substack{v\in C^{\infty},\ \Gamma \text{ smooth} \\ \Rightarrow \exists \eta\in\Gamma:}} - \frac{1}{2\pi a}\underbrace{\int_{\Gamma} v}_{\substack{v\in C^{\infty} \\ \Rightarrow \exists \xi\in\Gamma:}} =$$

$$\int_{\Gamma}\nabla v\cdot n = (\nabla v\cdot n)\big|_{\eta}\cdot\int_{\Gamma} ds \qquad \int_{\Gamma} v = v(\xi)\int_{\Gamma} ds$$

$$= \left\{\int_{\Gamma} ds = 2\pi a\right\} = \frac{2\pi a}{2\pi}\log\left(\frac{1}{a}\right)(\nabla v\cdot n)\big|_{\eta} - \frac{2\pi a}{2\pi a}v(\xi) =$$

$$= \underbrace{a\log\left(\frac{1}{a}\right)}_{\to 0,\ a\to 0^+}\underbrace{(\nabla v\cdot n)\big|_{\eta}}_{\text{bdd}} \to 0 \quad - v(\xi) \longrightarrow -v(0),\ a\to 0^+$$

lies on $|\xi| = a$

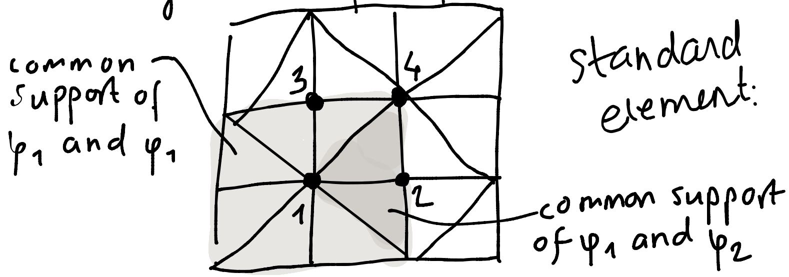
$$\Rightarrow -\int_{\mathbb{R}^2}\Delta E(x)v(x) \overset{\text{def.}}{=} -\int_{\mathbb{R}^2} E(x)\Delta v(x) = v(0)$$

any $v\in C_0^{\infty}$.

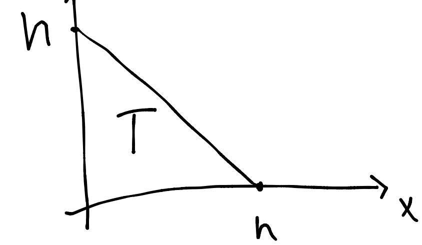
9.12) Formulate cG(1) method for the BVP

$$\begin{cases} -\Delta u + u = f & x\in\Omega \\ u = 0 & x\in\partial\Omega \end{cases}$$

Write down the matrix form of the resulting system of equations using the uniform mesh:




standard element:

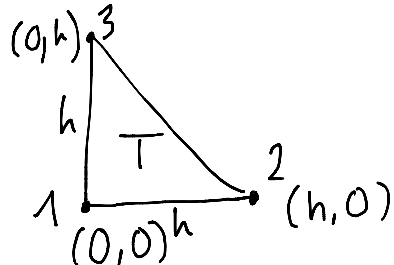


Solution. VF: $\displaystyle\int_\Omega \nabla u\cdot\nabla v + \int_\Omega uv = \int_\Omega fv \quad \forall v\in H_0^1(\Omega)$

FEM: Let $u_h = \displaystyle\sum_{j=1}^{4} \xi_j \varphi_j \quad \varphi_j =$ tent fcn w/ $\varphi_j(x_j) = 1$

$$\Rightarrow \sum_{j=1}^{4} \xi_j \Big( \underbrace{\int_\Omega \nabla\varphi_j \nabla\varphi_i}_{S=(S_{ij})} + \underbrace{\int_\Omega \varphi_i\varphi_j}_{M=(M_{ij})} \Big) = \int_\Omega f\varphi_i \qquad i=1,\dots,4$$

For the standard element $T$, we have:




area: $h^2/2$

$$\phi_1(x,y) = \frac{h-x-y}{h} \qquad \nabla\phi_1 = \left(-\frac{1}{h}, -\frac{1}{h}\right)$$

$$\phi_2(x,y) = \frac{x}{h} \qquad \nabla\phi_2 = \left(\frac{1}{h}, 0\right)$$

$$\phi_3(x,y) = \frac{y}{h} \qquad \nabla\phi_3 = \left(0, \frac{1}{h}\right)$$

Find local stiffness matrix, $S$:

$$\int_T \nabla\phi_1\cdot\nabla\phi_1 = \int_T \frac{2}{h^2} = \frac{h^2}{2}\cdot\frac{2}{h^2} = 1$$

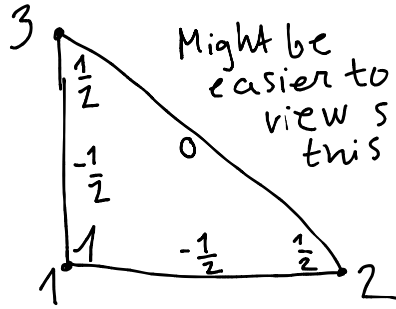
$$\int_T \nabla\phi_1\cdot\nabla\phi_2 = \dots = -\frac{1}{2}$$

$$\int_T \nabla\phi_2\cdot\nabla\phi_2 = \int_T \nabla\phi_3\cdot\nabla\phi_3 = \dots = \frac{1}{2}$$

$$\int_T \nabla\phi_2\cdot\nabla\phi_3 = \int_T 0 = 0$$

$$S = \begin{bmatrix} \int \nabla\varphi_1 \nabla\varphi_1 & \cdots \\ \vdots & \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$S =$ 


Might be easier to view s this way

Find local mass matrix, m:

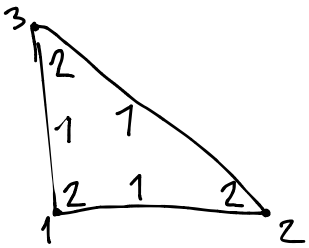
$$\int_T \phi_1 \phi_1 = \frac{h^2}{12}$$

$$\int_T \phi_1 \phi_2 = \int_T \phi_2 \phi_1 = \frac{h^2}{24}$$

$$\int_T \phi_2 \phi_2 = \int_T \phi_3 \phi_3 = \frac{h^2}{12}$$

$$\int_T \phi_2 \phi_3 = \int_T \phi_3 \phi_2 = \frac{h^2}{24}$$

$$m = \frac{h^2}{24} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$m = \frac{h^2}{24}$ 


Assemble global matrices:

$$S = \begin{bmatrix} 8 \cdot s_{22} & 2 \cdot s_{12} & 2 \cdot s_{12} & 2 \cdot s_{23} \\ 2 \cdot s_{12} & 4 \cdot s_{11} & 0 & 2 \cdot s_{12} \\ 2 \cdot s_{12} & 0 & 4 s_{11} & 2 s_{12} \\ 2 s_{23} & 2 s_{12} & 2 s_{12} & 8 s_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}$$

$$M = \begin{bmatrix} 8 \cdot m_{22} & 2 \cdot m_{12} & 2m_{12} & 2m_{23} \\ 2m_{12} & 4m_{11} & 0 & 2m_{12} \\ 2m_{12} & 0 & 4m_{11} & 2m_{12} \\ 2m_{23} & 2m_{12} & 2m_{12} & 8m_{22} \end{bmatrix}$$

$$= \frac{h^2}{24} \begin{bmatrix} 16 & 2 & 2 & 2 \\ 2 & 8 & 0 & 2 \\ 2 & 0 & 8 & 2 \\ 2 & 2 & 2 & 16 \end{bmatrix}$$

Resulting system: $(S+M)\xi = b$

$$b = \left[ \int_\Omega f\varphi_1 \ \int_\Omega f\varphi_2 \ \int_\Omega f\varphi_3 \ \int_\Omega f\varphi_4 \right]^T.$$

37) Describe a discrete system of eqns for a pw polynomial approx. for

$$\begin{cases} -\Delta u(x) = f(x) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases}$$

with cont. polyn. basis fcns $\{\varphi_j\}_{j=1}^M$.

Show that the stiffness matrix is symmetric and positive definite.

<u>Solution.</u> Let $V_h = \{v: \text{p.w. polyn, cont}, v=0 \text{ on } \Gamma\}$

$= \text{span}\{\varphi_j\}_{j=1}^M$

Discrete VF of problem:

Find $u_h \in V_h$ s.t $(\nabla u_h, \nabla v) = (f, v)$ $\forall v \in V_h$

Let $u_h = \sum_{j=1}^{M} \xi_j \varphi_j$. Then probl. is:

Find $\xi = (\xi_j)_{j=1}^{M} \in \mathbb{R}^M$ s.t

$$\sum_{j=1}^{M} \xi_j (\nabla \varphi_j, \nabla \varphi_i) = (f, \varphi_i) \quad \text{for } i = 1, \dots, M$$

$$A\xi = b, \quad A = (a_{ij})_{i,j=1}^{M} \quad a_{ij} = (\nabla \varphi_j, \nabla \varphi_i)$$

$$b = (b_i)_{i=1}^{M} \quad b_i = (f, \varphi_i)$$

A is obviously symmetric

Positive def.: $v^T A v > 0$, $v \neq 0$

Take $v \in \mathbb{R}^M \setminus \{0\}$. $w = \sum_{i=1}^{M} v_i \varphi_i \in V_h \setminus \{0\}$

$$v^T A v = \sum_{i=1}^{M} \sum_{j=1}^{M} v_i (\nabla \varphi_j, \nabla \varphi_i) v_j =$$

$$= \left( \sum_{j=1}^{M} v_j \nabla \varphi_j, \sum_{i=1}^{M} v_i \nabla \varphi_i \right) =$$

$$= \left( \nabla \sum_{j=1}^{M} v_j \varphi_j, \nabla \sum_{i=1}^{M} v_i \varphi_i \right) =$$

$$= (\nabla w, \nabla w) = \|\nabla w\|_{L^2}^2 \geq 0$$

$v^T A v = 0 \iff \nabla w = 0 \iff w$ const.

$\left\{\begin{array}{l} w \equiv 0 \text{ on } \partial\Omega \\ w \text{ cont.} \end{array}\right\} \Leftrightarrow w \equiv 0 \Leftrightarrow v = 0$

Thus, $v^T A v > 0$ for $v \neq 0$ and thus $A$ is pos. def. $\therefore$

38) Def. $P_h v$ ($L^2$-proj.) of $v \in L^2$ into the FE space by

$$(P_h v, w)_{L^2} = (v, w)_{L^2} \quad \forall w \in V_h$$

Also def. discrete laplacian $\Delta_h$ by $-(\Delta_h w, v) = (\nabla w, \nabla v) \quad \forall v \in V_h$.

Verify that we may express

$$(\nabla U, \nabla v) = (f, v) \quad \forall v \in V_h$$

as finding $U \in V_h$ s.t.

$$-\Delta_h U = P_h f$$

Solution

$$-\Delta_h U = P_h f \text{ in } V_h$$

$$\Leftrightarrow \quad -(\Delta_h U, v) = (P_h f, v) \quad \forall v \in V_h$$

$$\overset{\text{def } \Delta_h}{\Leftrightarrow} \quad (\nabla U, \nabla v) = (P_h f, v) \quad \forall v \in V_h$$

$$\overset{\text{def}^{\text{n}}}{\Longleftrightarrow} \quad (\nabla U, \nabla v) = (f, v) \qquad \forall v \in V_h.$$