Recall:

- Poisson in $\Omega \subset \mathbb{R}^2$ with polygonal $\partial\Omega$:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

- (VF) Find $u \in H_0^1(\Omega)$ s.t. $\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \quad \forall v \in H_0^1(\Omega)$

- (FE) Find $u_h \in V_h^0$ s.t. $\int_\Omega \nabla u_h \cdot \nabla \chi = \int_\Omega f \chi \quad \forall \chi \in V_h^0$

- Stiffness matrix S: $\int_\Omega \nabla \varphi_i \cdot \nabla \varphi_j \, dx$

$$S = \int_\Omega \begin{pmatrix} \nabla\varphi_1 \cdot \nabla\varphi_2 & \nabla\varphi_2 \nabla\varphi_1 & & * \; * \\ \nabla\varphi_2 \cdot \nabla\varphi_1 & \nabla\varphi_2 \nabla\varphi_2 & \ddots & * \; * \\ * & & \ddots & \\ * & * & & \nabla\varphi_5 \nabla\varphi_5 \end{pmatrix} =$$

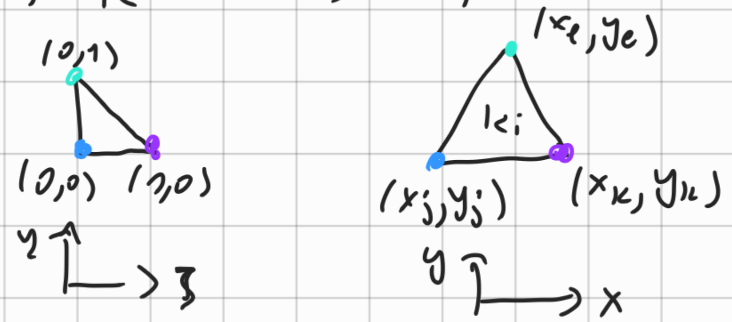
$$= \int_{K_1} \begin{pmatrix} \nabla\varphi_1 \nabla\varphi_1 & 0 & 0 & \nabla\varphi_4 \nabla\varphi_1 & \nabla\varphi_5 \nabla\varphi_1 \\ 0 & \cdots & & & \cdots \end{pmatrix} + \int_{K_2} \begin{pmatrix} \nabla\varphi_1 \nabla\varphi_1 & \nabla\varphi_2 \nabla\varphi_1 & 0 & \nabla\varphi_4 \nabla\varphi_1 & 0 \\ \cdots \end{pmatrix}$$

$$+\int_{K_3} \underbrace{\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}}_{S_3 \text{ element stiffness matrix } (3\times 3)}$$

c) Computation of element stiffness/mass matrix:

To compute the element stiffness matrix, one introduces a linear map:

$$\Phi : \hat{K} \longrightarrow K_i$$



$$(0,0) \longmapsto (x_j, y_j)$$

$$(1,0) \longmapsto (x_k, y_k)$$

$$(0,1) \longmapsto (x_\ell, y_\ell)$$

We use the hat functions on the reference triangle $\hat{K}$ in order to define Φ.

Recall that these hat functions / element shape functions are give by

$$\hat{\varphi}_1(\xi,\eta) = 1-\xi-\eta$$

$$\hat{\varphi}_2(\xi,\eta) = \xi$$

$$\hat{\varphi}_3(\xi,\eta) = \eta$$

The map $\Phi(\xi,\eta)$ is then defined by

$$\Phi(\xi,\eta) = \begin{pmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{pmatrix} = \begin{pmatrix} x_j \\ y_j \end{pmatrix} \hat{\varphi}_1(\xi,\eta) + \begin{pmatrix} x_k \\ y_k \end{pmatrix} \hat{\varphi}_2(\xi,\eta) + \begin{pmatrix} x_\ell \\ y_\ell \end{pmatrix} \hat{\varphi}_3(\xi,\eta)$$

⚠ Now any integrals on the element/triangle K_i can be performed on the reference triangle $\hat{K}$ using the change of variables

$$(\xi,\eta) \overset{\Phi}{\longmapsto} (x,y) :$$

$$\int_{K_i} g(x,y)\,dx\,dy = \int_{\hat{K}} g(\Phi(\xi,\eta))\,|J|\,d\xi\,d\eta,$$

where $|J| = |\det(J)|$, with J the Jacobian of the linear map Φ:

$$J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \dots = \begin{pmatrix} x_k - x_j & x_\ell - x_j \\ y_k - y_j & y_\ell - y_j \end{pmatrix}$$

2x2 matrix that only depends on nodes of K_i!

Rem: • One can also write

$$\Phi(\xi, \eta) \underset{\uparrow}{=} \begin{pmatrix} x_j \\ y_j \end{pmatrix} + J \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

Def $\hat{\varphi}_j$

• $|K_i| = \text{Area}(K_i) = \dfrac{\text{Area}\left(\square\right)}{2} = \dots =$

$$= \frac{1}{2} \det(J) = \frac{1}{2}|J| \Rightarrow |J| = 2|K_i|$$

Illustration:

Compute $\displaystyle\int_{K_i} \nabla\varphi_j \cdot \nabla\varphi_k \, dx\,dy$:

$$\int_{K_i} \nabla\varphi_j \cdot \nabla\varphi_k \, dx\,dy \underset{\uparrow}{=} \int_{K_i} \nabla\varphi_j^T \nabla\varphi_k \, dx\,dy =$$

Def dot product

$$= \int_{\hat{K}} \left(\nabla \hat{\varphi}_1^T J^{-1}\right)\left(\nabla \hat{\varphi}_2^T J^{-1}\right)^T |J| \, d\xi \, d\eta =$$

Change of variables $(x,y) \leftrightarrow (\xi,\eta)$

$\hat{\varphi} = \varphi \circ \Phi$

+ chain rule : $\nabla \hat{\varphi}^T = \nabla \varphi^T J$

$$= \int_{\hat{K}} \left(\nabla \hat{\varphi}_1^T \underbrace{J^{-1} J^{-T}}_{\text{depends on nodes of } K_i} \nabla \hat{\varphi}_2\right) |J| \, d\xi \, d\eta =$$

$\nabla \hat{\varphi}_1^T = (-1,-1)$, $\nabla \hat{\varphi}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|J| = 2 \cdot |K_i|$

$\hat{\varphi}_1 = 1 - \xi - \eta$

$$= (-1,-1) J^{-1} J^{-T} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot 2 \cdot |K_i| \cdot \underbrace{\int_{\hat{K}} d\xi \, d\eta}_{\frac{1}{2} \ (|\hat{K}|)}$$

$$= |K_i| \, (-1,-1) J^{-1} J^{-T} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad //$$

d) Computation of the load vectors

This is done as above for the stiffness matrix, by assembling each 3×1 element load vectors:

$$(b_K)_{i=1}^{3} = \left(\int_K f(x,y)\,\varphi_i(x,y)\,dx\,dy \right)_{i=1}^{3}$$

Let us use a simple quadrature formula to approximate these integrals:

$$\int_K f(x,y)\,\varphi_i(x,y)\,dx\,dy \approx f(N_i) \int_K \varphi_i(x,y)\,dx\,dy \approx$$

$f(x,y) \approx f(N_i)$, where N_i is node i of K.

$$\approx f(N_i) \int_{\hat{K}} \hat{\varphi}_i \, |J| \, d\xi\, d\eta \approx 2\cdot|K|\cdot f(N_i) \underbrace{\int_{\hat{K}} \hat{\varphi}_i \, d\xi\, d\eta}_{\frac{1}{6} \text{ for } i=1,2,3}$$

go to the reference triangle $\hat{K}$

$|J| = 2\cdot|K|$

$\Rightarrow$ Finally, one has the following result:

$$b_K \approx \frac{1}{3}\cdot|K|\cdot \begin{pmatrix} f(N_1) \\ f(N_2) \\ f(N_3) \end{pmatrix}$$

, where N_1, N_2, N_3 are nodes of K.

3) A priori error estimates:

We start with

Th: (Poincaré inequality)

Let $\Omega \subset \mathbb{R}^2$ bounded domain with smooth $\partial\Omega$.

Then, $\exists$ $C>0$ s.t.

$$\|v\|_{L^2(\Omega)} \leq C \cdot \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega).$$

Proof:

- Consider a function Φ with the properties

$$-\Delta \Phi = 1 \text{ in } \Omega \quad \text{and} \quad \sup_{x \in \Omega} |\nabla \Phi(x)| \leq \tilde{C} < \infty$$

(see below for an example)

- Next, compute

$$\|v\|^2_{L^2(\Omega)} \underset{\text{Def } \|\cdot\|_{L^2}}{=} \int_\Omega v^2(x) \cdot 1 \, dx \underset{1 = -\Delta\Phi}{=} -\int_\Omega v^2(x) \Delta\Phi(x) dx \underset{\text{Green's formula}}{=}$$

$$= \int_\Omega \nabla(v^2)\cdot\nabla(\Phi)\,dx - \underbrace{\int_{\partial\Omega} v^2 n\cdot\nabla\Phi\,ds}_{=0 \text{ since } v\in H_0^1} =$$

$$= \int_\Omega (2v\nabla v)\cdot \underbrace{\nabla\Phi}_{\leq \sup_{x\in\Omega} |\nabla\Phi(x)| \leq \hat{C} < \infty}\,dx$$

$$\leq C\cdot\int_\Omega v\nabla v\,dx \underset{C-S}{\leq} C\cdot \|v\|_{L^2(\Omega)}\cdot\|\nabla v\|_{L^2(\Omega)}$$

Finally, $\|v\|_{L^2}^{\cancel{2}} \leq C\cdot \cancel{\|v\|_{L^2}}\cdot\|\nabla v\|_{L^2} \Rightarrow$

$$\Rightarrow \|v\|_{L^2} \leq C\cdot\|\nabla v\|_{L^2}$$

- One example of such $\Phi(x_1,x_2)$ is given by $\Phi(x_1,x_2) = -\frac{1}{4}(x_1^2+x_2^2)$:

$$\nabla\Phi(x_1,x_2) = -\frac{1}{2}(x_1,x_2)$$

$$\Delta\Phi(x_1,x_2) = -\frac{1}{2}-\frac{1}{2} = -1 \quad :-)$$

$$|\nabla\Phi(x_1,x_2)| = \sqrt{\left(\frac{1}{2}x_1\right)^2+\left(\frac{1}{2}x_2\right)^2} = \frac{1}{2}\sqrt{x_1^2+x_2^2}$$

For $(x_1,x_2)\in\Omega$, we get $|\nabla\Phi(x_1,x_2)| \le \frac{1}{2}\,\mathrm{diam}(\Omega)$

$\le C < \infty$.

$\blacksquare$

Next, we show

<u>Th</u>: (Galerkin's orthogonality (GO))

Let u, u_h the sol. to Poisson's eq, resp. its FE approximation, then

$$\int_\Omega \nabla(u-u_h)\cdot\nabla v\,dx = 0 \quad \forall\, v\in V_h^0$$

<u>Proof</u>:

(VF) reads
$$\int_\Omega \nabla u\cdot\nabla v\,dx = \int_\Omega f v\,dx \quad \forall\, v\in H_0^1$$

(FE) reads
$$\int_\Omega \nabla u_h\cdot\nabla v\,dx = \int_\Omega f v\,dx \quad \forall\, v\in V_h^0\subset H_0^1$$

Taking the difference provides

$$\int_\Omega \nabla(u-u_h)\cdot\nabla v\,dx = 0 \qquad \forall\, v\in V_h^0$$

Using the above, one shows that u_h is the best approximation of u in V_h^0 in the energy norm:

Th: (Best approximation)

Under the notations of (GO), one has

$$\|u-u_h\|_E \leq \|u-v\|_E \qquad \forall\, v\in V_h^0,$$

where we recall the energy norm

$$\|v\|_E = \|\nabla v\|_{L^2(\Omega)} \qquad \text{(for Poisson's eq.)}$$

Proof:

$$\|u-u_h\|_E^2 \underset{\uparrow}{=} \int_\Omega \nabla(u-u_h)\cdot\nabla(u-v+v-u_h)\,dx =$$

Def $\|\cdot\|_E$, $v\in V_h^0$

$$= \int_\Omega \nabla(u-u_h)\cdot\nabla(u-v)\,dx + \int_\Omega \nabla(u-u_h)\cdot\nabla(\underbrace{v-u_h}_{\in V_h^0})\,dx$$

$= 0$ by (GO)

$$= \int_\Omega \nabla(u-u_h)\cdot\nabla(u-v)\,dx \overset{C-S}{\le} \|u-u_h\|_E \, \|u-v\|_E$$

$\Rightarrow \|u-u_h\|_E \le \|u-v\|_E \quad \forall v \in V_h^0$ ∎

This is used to prove

Th: (a priori error estimates in energy norm)

For $u \in H^2(\Omega)$ sol. (VF) and u_h sol. to (FE), one has

$$\|u-u_h\|_E^2 \le C \cdot \sum_{K \in T_h} h_K^2 \cdot \|u\|_{H^2(K)}^2$$

Proof:

We have

$$\|u-u_h\|_E^2 \le \|u-\Pi_h u\|_E^2 = \|\nabla(u-\Pi_h u)\|_{L^2(\Omega)}^2$$

u_h best approx

$\Pi_h u \in V_h^0$ interpolant of u

$$\underset{\uparrow}{\leq} C \cdot \sum_{K \in T_h} h_K^2 \, \|u\|^2_{H^2(K)}$$

interpolation error, (Chapter X : 7)

Rem: • The above error estimate says more or less:

$$\|u - u_h\|_E \leq C \cdot h \, \|u\|_{H^2(\Omega)}$$

• How to measure the error in L^2?

$$\|u - u_h\|_{L^2(\Omega)} \underset{\substack{\uparrow \\ \text{Poincaré}}}{\leq} C \cdot \underbrace{\|\nabla(u - u_h)\|_{L^2(\Omega)}}_{\|u-u_h\|_E} \underset{\substack{\uparrow \\ \text{above remark}}}{\leq} C \cdot h \, \|u\|_{H^2(\Omega)}$$

• An optimal error estimate is given by

$$\|\underset{\substack{\uparrow \\ c(h/1)}}{u_h} - u_h\|_{L^2(\Omega)} \leq C \cdot h^{\boxed{2}} \, \|u\|_{H^2(\Omega)}$$